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**EQUILIBRIA WITHOUT THE SURVIVAL  
ASSUMPTION: A NON-STANDARD ANALYSIS  
APPROACH**

By Alexander Konovalov and Valeri Marakulin

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Discussion paper

# Equilibria without the survival assumption: a non-standard analysis approach

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## Abstract

It is well known that an equilibrium in the Arrow-Debreu model may fail to exist if a very restrictive condition called the survival assumption is not satisfied. We study two approaches that allow for the relaxation of this condition. Danilov and Sotskov (1980), and Florig (1998a) developed a concept of a generalized equilibrium based on a notion of hierarchic prices. Marakulin (1988, 1990) proposed a concept of an equilibrium with non-standard prices. In this paper, we establish the equivalence between non-standard and hierarchic equilibria. Furthermore, we show that the set of non-standard equilibria coincides with the fuzzy rejective core of an economy and is generically finite for any specified system of dividends.

## 1 Introduction

It is well known that an equilibrium in the Arrow-Debreu model may fail to exist if a very restrictive condition called the survival assumption is not satisfied. Its most widely used and widely criticized version requires that every consumer have a positive initial endowment of every good existing in the economy.

To illustrate the problem consider an example (cf. Gale (1976)) of a market with two traders and two commodities: apples and oranges. The first trader owns apples and oranges, but has a positive utility only for apples, the

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second trader cares for both, but owns only oranges. If the price of oranges is positive then the first agent sells his oranges in order to buy more apples, but he already has all the apples. If prices of oranges is zero then the second agent demands an infinite amount of oranges. Thus, no equilibrium results. The reason for this is that the second trader's budget correspondence is not lower hemicontinuous. As the price for oranges falls to zero, the budget set and the demand "explode".

An idea that suggests itself is to redefine a budget correspondence by appropriate refining the notion of prices in order to get an equilibrium that always exists. In particular, an equilibrium in Gale's example would be restored if one manages to define prices for oranges so small that no apples can be bought for any amount of oranges, but still non-zero.

Two realizations of this idea were proposed so far. Gay (1978), Danilov and Sotskov (1990), Mertens (1996), and Florig (1998a, 1998b) developed an approach based on a notion of a hierarchic price. At equilibrium, all commodities (or commodity bundles treated as separate goods) are divided into several disjoint classes and traded against commodities of the same class according to prices which are an element of some set called a hierarchic price. Moreover, the set of such classes is ordered, superior class commodities cost infinitely much compared to the inferior class ones.

Marakulin (1988, 1990) uses non-standard prices in the sense of Robinson's infinitesimal analysis (Robinson (1966)). A similar hierarchic structure of submarkets arises.

An idea to use non-standard numbers to measure prices may look odd at first sight. But a second thought shows that it is not a much bigger abstraction than the use of real numbers for this purpose. Hardly anyone ever paid to anyone else a price of  $\sqrt{2}$ . Besides, non-standard prices are even natural, since they reflect the fact that costs and values (which are no more than mere numbers) are usually "more divisible" than quantities of consumption goods such as cars, houses, pieces of clothing, etc. The only disadvantage of this approach seems to be that there are still relatively few working economists trained in non-standard analysis. On the other hand, it clearly exceeds standard ways in elegance of proofs and generality of results.

The first contribution of the present paper is to reconcile the standard and non-standard approaches. Starting with non-standard equilibrium, we derive a unique representation of non-standard prices by a hierarchic price, which allows us to characterize non-standard budget sets in pure standard terms. The equivalence between non-standard equilibria and Florig's hierarchic equilibria follows. Next, we prove that the set of non-standard equilibria coincides with the fuzzy rejective core of an economy, a concept introduced in Konovalov (1998). Another important result presented in the paper is the

representation of the set of all hierarchic prices by a union of manifolds of dimension less than  $l - 1$ , where  $l$  is a number of consumption goods in an economy. Using this result, we show that the number of competitive equilibria with non-standard prices is generically finite if agents' utility functions are sufficiently smooth.

To summarize, the three main results presented in the paper include 1. the equivalence between the set of non-standard dividend equilibria and the set of hierarchic equilibria, 2. the equivalence of the set of non-standard dividend equilibria to the fuzzy rejective core of an economy, and 3. the generic finiteness of the set of non-standard dividend equilibria.

Section 2 provides a reader with a definition of an equilibrium with non-standard prices and an example which motivates the use of this concept. The next section contains a number of auxiliary results that allow us to describe the set of non-standard equilibria in pure standard terms and establish the equivalence of non-standard and hierarchic equilibria. In Section 4 we study the set of hierarchic equilibrium prices. Section 5 is devoted to optimality and core equivalence properties of non-standard equilibria. In Section 6 we prove the existence of non-standard dividend equilibria for any specified system of dividends. Section 7 contains the finiteness result.

## 2 Equilibrium with non-standard prices

We work with an exchange economy  $\mathcal{E}$  defined by

$L = \{1, \dots, l\}$  — the set of commodities;

$Q \subseteq \mathbb{R}^l$  — the set of admissible prices;

$N = \{1, \dots, n\}$  — the set of agents, where each agent  $i \in N$  is characterized by his consumption set  $X_i \subset \mathbb{R}^l$ , initial endowments  $w^i \in X_i$  and preferences given by a correspondence  $\mathcal{P}_i : X_i \rightarrow 2^{X_i}$ ,  $\mathcal{P}_i(x^i)$  denotes the set of consumption bundles *strictly* preferred to  $x^i$ .

Consider the  $*$ -image  $*Q$  of the set  $Q$  as the set of all admissible non-standard prices and define by analogy with the standard case non-standard budget sets of consumers.

$$*B_i(p) = \{x \in *X_i \mid px \leq pw^i\}, \quad p \in *Q, \quad i \in N.$$

By definition, these sets consist of non-standard consumption plans. Consider their standard parts  $\bar{B}_i(p)$  :

$$\bar{B}_i(p) = st *B_i(p) = \{x \in X_i \mid \exists \tilde{x} \in *B_i(p) : \tilde{x} \approx x\},$$

where  $\tilde{x} \approx x$  denotes infinitesimality of the difference  $\tilde{x} - x$  :  $\|\tilde{x} - x\| \approx 0$ . An equilibrium with non-standard prices is formally defined by substitution

of the set of possible prices and budget sets in the traditional notion of equilibrium for  ${}^*Q$  and  $\bar{B}_i(p)$  respectively.

**Definition 1** An allocation  $\bar{x} \in \prod_{i \in N} X_i$  is an *equilibrium with non-standard prices* if there exist  $p \in {}^*Q$  such that the following conditions hold:

- (i) attainability:  $\bar{x}^i \in \bar{B}_i(p)$ ,  $i \in N$ ,
- (ii) individual rationality:  $\mathcal{P}_i(\bar{x}^i) \cap \bar{B}_i(p) = \emptyset$ ,  $i \in N$ ,
- (iii) market clearing:  $\sum_{i \in N} \bar{x}^i = \sum_{i \in N} w^i$ .

It readily follows from the definition above that each Walrasian equilibrium is also an equilibrium with non-standard prices.

**Proposition 1** *Suppose that  $\bar{x}$  is a Walrasian equilibrium allocation and  $\bar{p} \in Q$  is a corresponding standard vector of competitive prices. Then  $\bar{x}$  is an equilibrium with non-standard prices.*

**Proof.** Take  $\bar{p} \in Q \subset {}^*Q$  as a non-standard equilibrium price vector. Consider an arbitrary individual  $i$ . A Walrasian budget set

$$B_i(\bar{p}) = \{x \in X_i | \bar{p}x \leq \bar{p}w^i\}$$

is by definition a subset of  $\bar{B}_i(\bar{p})$ . To prove the proposition, we need to show that  $\bar{B}_i(\bar{p}) \subseteq B_i(\bar{p})$ . Let  $x \in \bar{B}_i(\bar{p})$ , then there exists  $\tilde{x} \in {}^*B_i(\bar{p})$  infinitely near to  $x$ . Since  $B_i(\bar{p})$  is closed in  $X_i$ ,  $x \in B_i(\bar{p})$  by the non-standard condition for closedness (see Anderson (1991), Proposition 2.2.2). Therefore,  $B_i(\bar{p}) = \bar{B}_i(\bar{p})$  for every  $i$ , which implies that  $\bar{x}$  is a Walrasian equilibrium allocation.  $\square$

Reversely, an equilibrium with non-standard prices  $p$  is a Walrasian equilibrium if the survival condition is satisfied for each agent  $i$  at prices  $\bar{p} = {}^\circ(p/\|p\|)$ . Recall that  ${}^\circ a$  for  $a \in {}^*\mathbb{R}^l$  denotes a *standard part* of  $a$ , that is an element of  $\mathbb{R}^l$  such that  ${}^\circ a \approx a$ .

**Proposition 2** *Suppose that  $\bar{x}$  is a non-standard equilibrium with non-standard prices  $p \in {}^*Q$ . Let  $\bar{p} = {}^\circ(p/\|p\|)$ . If*

$$\inf \bar{p}X_i < \bar{p}w^i, \quad i \in N,$$

*then  $\bar{x}$  is a Walrasian equilibrium sustained by the price system  $\bar{p}$ .*

The proof of this proposition is relegated to an appendix since it uses the facts that appear later in the paper.

Replacing (iii) in the Definition 1 by

$$\sum_{i \in N} \bar{x}^i \leq \sum_{i \in N} w^i,$$

gives a definition of a *semi-equilibrium with non-standard prices*.

## A satiation effect and dividends

Unfortunately, non-standard equilibria typically do not exist due to a satiation effect caused by measuring prices in non-standard numbers. Recall that an agent is said to be satiated if his demand does not belong to the boundary of his budget set. If there is at least one such an agent in an economy, Walras' law is violated and no competitive equilibrium exists. In our model, even if agent  $i$ 's preferences are locally non-satiated, his demand belongs to the boundary of the standard set  $\bar{B}_i(p)$  but not necessarily to the boundary of the set  ${}^*B_i(p)$ , which almost entirely consists of non-standard points. In other words, infinitesimal budget excess may be created, which results in possible non-standard equilibrium existence failure.

The problem is alike the one caused by indivisibilities. Suppose that a smallest available quantity of the good that I need is  $\varepsilon$  and its price is  $p_1$ . If the amount  $\delta$  of free value at my disposal is less than  $p_1\varepsilon$ , I can not use it to increase my utility. All the more, if  $\delta$  is infinitesimal while  $p_1$  is not, no standard quantity of such a good is achievable. In any case, the consequence of this infinitely small value being unused is an inequality  $\sum_{i \in N} p\bar{x}^i < \sum_{i \in N} pw^i$ , where  $\bar{x}^i$  are individual demands, which is inconsistent with the equilibrium market clearing condition.

Semi-equilibria with non-standard prices exist if all consumption sets are positive orthants (Marakulin (1990)). In a more general case, where  $X_i$  are convex closed bounded from below sets, there exist dividend equilibria with non-standard prices. A notion of dividend equilibrium was proposed by several authors (Makarov (1981), Aumann and Drèze (1986), Mas-Colell (1992)) in order to analyse economies which allowed for possibly satiated preferences. In a dividend equilibrium, each agent  $i$ 's budget constraint is relaxed by some slack variable  $d_i$  in order to allow for redistribution of a budget excess created by satiated agents among non-satiated ones. Such a slack variable can be interpreted as an agent's endowment of coupons (as in Drèze and Müller (1980)) or paper money (as in Kajii (1996)).

For prices  $p \in {}^*Q$  and a *system of dividends*  $d \in {}^*\mathbb{R}_+^n$  consider the sets

$$\bar{B}_i(p, d_i) = st \{x \in {}^*X_i \mid px \leq pw^i + d_i\}, \quad i \in N.$$

**Definition 2** An allocation  $\bar{x} \in X$  is a *non-standard dividend equilibrium*, if there exist non-standard vectors  $d = (d_1, \dots, d_n) \in {}^*\mathbb{R}_+^n$  and  $p \in {}^*Q$ , such that the following conditions hold:

- (i) attainability:  $\bar{x}^i \in \bar{B}_i(p, d_i)$ ,  $i \in N$ ,
- (ii) individual rationality:  $\mathcal{P}_i(\bar{x}^i) \cap \bar{B}_i(p, d_i) = \emptyset$ ,  $i \in N$ .

(iii) market clearing:  $\sum_{i \in N} \bar{x}^i = \sum_{i \in N} w^i$ .

It is well-known that if agents' preferences are locally non-satiated, then all dividend terms at a dividend equilibrium are equal to zero, which makes it a usual Walrasian equilibrium. Similarly, one can assert that if  $\bar{x}$  is a non-standard dividend equilibrium and local non-satiation of preferences holds, then every  $d_i$  is infinitesimal.

**Proposition 3** *Suppose that  $\bar{x}$  is a non-standard dividend equilibrium such that an equilibrium price vector  $p$  is nearstandard. Assume that for every  $\varepsilon \in \mathbb{R}_{++}$ , for all  $i \in N$ , and every  $x^i \in X_i$  there exists  $y^i \in \mathcal{P}_i(x^i)$  such that  $\|y^i - x^i\| < \varepsilon$ . Then  $d_i \approx 0$  for every  $i \in N$ .*

Proof. Consider the standard part  ${}^\circ p$  of an equilibrium price vector  $p$  and show that

$${}^\circ p \bar{x}^i = {}^\circ p w^i + {}^\circ d_i, \quad i \in N. \quad (1)$$

The inequality  ${}^\circ p \bar{x}^i > {}^\circ p w^i + {}^\circ d_i$  would imply that  $\bar{x}^i$  is not attainable. Suppose that

$${}^\circ p \bar{x}^i < \tau < {}^\circ p w^i + {}^\circ d_i,$$

for some  $\tau \in \mathbb{R}$ . Then there exists  $\varepsilon \in \mathbb{R}_{++}$  such that  ${}^\circ p y^i < \tau < {}^\circ p w^i + {}^\circ d_i$  for every  $y^i \in \{z^i \in X_i \mid \|x^i - z^i\| < \varepsilon\}$ . By continuity of scalar multiplication,  $p \tilde{y}^i < \tau < p w^i + d_i$  for every  $\tilde{y}^i \approx y^i$ . Therefore, every element of some standard neighborhood of  $\bar{x}^i$  is also an element of the budget set  $\bar{B}_i(p, d_i)$ . By the local non-satiation condition, an intersection  $\bar{B}_i(p, d_i) \cap \mathcal{P}_i(\bar{x}^i)$  is not empty, which is a contradiction with the individual rationality of  $\bar{x}^i$ . Therefore (1) is true. To complete the proof, one needs to observe that all  ${}^\circ d_i$  have to be equal to zero if (1) is to be consistent with the equilibrium market clearing condition.

□

The need for using non-standard prices and dividends is illustrated by the following example (cf. Florig (1998)). In this example not only Walrasian equilibria, but also equilibria and semi-equilibria with non-standard prices fail to exist. However, a non-standard dividend equilibrium exists.

Consider a market with two agents and two goods: white bread and brown bread. Each agent requires a minimum of four slices of bread a day to survive and has an initial endowment of three slices of white bread and one slice of brown bread. The first consumer likes only white bread, and the second consumer likes only brown bread. Formally, consumption sets and initial endowments of agents are given by  $X_1 = X_2 = \{x \in \mathbb{R}_+^2 : x_W + x_B \geq$



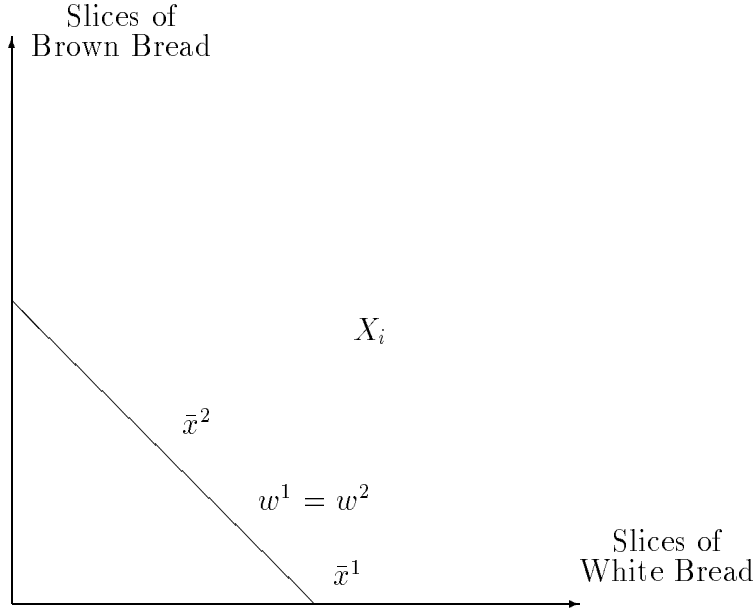


Figure 1: Non-existence of equilibria without non-standard prices and dividends.

4},  $w^1 = w^2 = (3, 1)$ , their preferences are represented by utility functions  $u_1(x) = x_W, u_2(x) = x_B$ .

An allocation  $\bar{x}^1 = (4, 0), \bar{x}^2 = (2, 2)$  looks especially attractive, in fact, it is the only Pareto optimum. Nevertheless, it can not be obtained through the market mechanism unless non-standard prices and dividends are employed. If prices are different from  $(1, 1)$ , then one of two agents demands more than 4 slices of bread, which would push the other agent outside of his consumption set. Therefore, neither equilibrium nor semi-equilibrium is possible. If  $p = (1, 1)$  then excess demand for brown bread is greater than zero, the same is true for non-standard prices  $p = (1 + \varepsilon, 1 - \varepsilon)$ , where  $\varepsilon$  is some positive infinitesimal. For prices  $p = (1 - \varepsilon, 1 + \varepsilon)$  the excess demand for white bread exceeds zero. By Proposition 2, the case  $^{\circ}p = (1, 1)$  is the only one where non-standard prices matter. Thus, the set of Walrasian equilibria, as well as the sets of equilibria and semi-equilibria with non-standard prices, are empty for this economy. However,  $\bar{x}$  is a non-standard dividend equilibrium

if prices are  $p = (1 - \varepsilon, 1 + \varepsilon)$  and dividends are  $d = (0, 2\varepsilon)$ . What is happening is that in addition to the standard price system  $(1, 1)$  a commodity bundle  $(-1, 1)$  is priced by an infinitesimal value encouraging the sale of brown bread and the purchase of white bread. One could say that infinitesimal prices and dividends determine an additional constraint, which plays a role of a rationing scheme. Namely, suppose that  $d_i$  represents an agent  $i$ 's endowment of coupons. Then the second agent spends his amount of coupons  $2\varepsilon$  buying one slice of brown bread and selling one slice of white bread, and this is as much as he can achieve. The first agent does not have to spend any coupons at all.

### 3 Hierarchic prices and non-standard budget sets

In this section we study non-standard prices and budget sets. Among the results displayed are the representation of non-standard prices by an orthonormal set of standard vectors and characterization of non-standard dividend budget sets in pure standard terms. Throughout this section, we use simplified notations, in what follows  $X \subset \mathbb{R}^l$  denotes a consumption set of some individual. Without loss of generality we assume that his initial endowments are zero (the general situation can easily be reduced to this case by a shift of the consumption set). In the beginning of the section we deal with the following budget sets

$$B(p) = \{x \in X : px \leq 0\},$$

— a standard budget set for  $p \in \mathbb{R}^l$ ;

$${}^*B(p) = \{x \in {}^*X : px \leq 0\},$$

— a non-standard budget set for  $p \in {}^*\mathbb{R}^l$ ;

$$\bar{B}(p) = st {}^*B(p) = \{x \in X : \exists \tilde{x} \in {}^*X : \tilde{x} \approx x \text{ and } p\tilde{x} \leq 0\},$$

— a standardization of a non-standard budget set.

An ordered orthonormal set  $\{q_1, \dots, q_k\}$  of non-zero vectors in  $\mathbb{R}^l$  is called a *hierarchic price*. The first proposition projects the set of non-zero non-standard prices  ${}^*\mathbb{R}^l \setminus \{0\}$  onto the set of all possible hierarchic prices  $\cup_{k=1}^l V_{l,k}$ , where  $V_{l,k}$  is a Stiefel manifold.

**Proposition 4** For each  $p \in {}^*\mathbb{R}^l \setminus \{0\}$ , there exists an orthonormal set of standard vectors  $\{q_1, q_2, \dots, q_k\} \in \mathbb{R}^{lk}$  such that  $p$  has a unique representation

$$p = \lambda_1 q_1 + \dots + \lambda_k q_k, \quad (2)$$

with positive coefficients  $\lambda_j \in {}^*\mathbb{R}_{++}$  satisfying

$$\frac{\lambda_{j+1}}{\lambda_j} \approx 0, \quad j \in \{1, \dots, k-1\}. \quad (3)$$

**Proof.** The existence part of this proposition was proved in Marakulin (1988). In fact, for each  $p$  it is possible to express the components  $q_j$  of a corresponding hierarchic price and the coefficients  $\lambda_j$  explicitly. Namely,

$$q_1 = {}^\circ\left(\frac{p}{\|p\|}\right), \quad \lambda_1 = pq_1.$$

and

$$q_m = {}^\circ\left(\frac{v_m}{\|v_m\|}\right), \quad \lambda_m = pq_m,$$

where  $v_m = p - \sum_{j=1}^{m-1} \lambda_j q_j$ ,  $m = 2, \dots, k$ .

Suppose that  $\lambda_1 q_1 + \dots + \lambda_k q_k$  and  $\lambda'_1 q'_1 + \dots + \lambda'_m q'_m$  are two different representations of the same non-standard vector  $p$ . Then

$$0 = \sum_{j=1}^k \lambda_j q_j - \sum_{j=1}^m \lambda'_j q'_j. \quad (4)$$

Assume without loss of generality that  $m < k$ . Let  $\lambda_1$ , for example, be the largest element in the set  $\{\lambda_1, \dots, \lambda_k, \lambda'_1, \dots, \lambda'_m\}$ . Since all  $\lambda_j, \lambda'_j, j > 1$ , are infinitely smaller than  $\lambda_1$ , it must be that  $\lambda_1 q_1 = \lambda'_1 q'_1$ , which implies (recall  $\|q_1\| = \|q'_1\| = 1$ ,  $\lambda_1, \lambda'_1 > 0$ ) that  $\lambda_1 = \lambda'_1$ ,  $q_1 = q'_1$ . Similarly, taking the largest element of the set  $\{\lambda_2, \dots, \lambda_k, \lambda'_2, \dots, \lambda'_m\}$  one will come to  $\lambda_2 = \lambda'_2$ ,  $q_2 = q'_2$ , and so on, until  $\lambda_m = \lambda'_m$  and  $q_m = q'_m$ . But then (4) implies  $m = k$ .

□

Example:  $(1 + \varepsilon - \varepsilon^2, \varepsilon - 2\varepsilon^2, -\varepsilon^2) = (1 + \varepsilon - \varepsilon^2)(1, 0, 0) + (\varepsilon - 2\varepsilon^2)(0, 1, 0) + \varepsilon^2(0, 0, -1)$ .

If the relations (2)–(3) are true, we say that a hierarchic price  $(q_1, \dots, q_k)$  represents a non-standard price vector  $p$  and denote it by  $q^p$ . Conversely, if  $q = (q_1, \dots, q_k)$  is a hierarchic price, one may consider for some  $\varepsilon \approx 0, \varepsilon > 0$  a vector of non-standard prices

$$p(q, \varepsilon) = q_1 + \varepsilon q_2 + \dots + \varepsilon^{k-1} q_k$$

such that  $q^{p(q,\varepsilon)} = q$ . Evidently, there is much more than one non-standard vector that satisfies this property, in particular  $q^{p(q,\varepsilon)} = q^{p(q,\varepsilon^2)}$ . Furthermore, if  $p = 0$ , then put  $q^p = \{0\}$ .

The next proposition gives a characterization of the set  $\bar{B}(p)$  and is crucial for our analysis. It asserts that if  $X$  is a polyhedral set then there is a number  $m \in \{1, \dots, k\}$  such that the set  $\bar{B}(p)$  consists of elements  $x$  such that the vector  $(q_j x)_{j=1, \dots, m}$  is lexicographically less than zero.

Remind that a subset  $P \subset \mathbb{R}^l$  is called a *polyhedral set* or a *polyhedron* if it is the intersection of a finite number of closed half-spaces. Each polyhedral set is closed and convex but not necessarily compact.

Denote the set  $\{x \in X \mid q_1 x = 0, \dots, q_m x = 0\}$  by  $X(q_1, \dots, q_m)$  and put  $X(\emptyset) = X$ . Consider the sets

$$B_m(p) = \{x \in X(q_1, \dots, q_{m-1}) \mid q_m x \leq 0\}, \quad m \leq k, \quad (5)$$

and

$$B_{k+1}(p) = X(q_1, \dots, q_k). \quad (6)$$

**Proposition 5** *Suppose that  $p \in {}^*\mathbb{R}^l$ , and at least one of the following alternatives is true:*

- (a) *The set  $X$  is a polyhedron;*
- (b) *The set  $X$  is closed, star-shaped with respect to 0 and locally polyhedral at 0, that is for some small positive  $\gamma \in \mathbb{R}_{++}$  the set  $P = \{x \in X : |x_j| < \gamma, j \in L\}$  is a polyhedron.*

*Then there exists a natural number  $m \in \{1, \dots, k+1\}$  such that*

$$\bar{B}(p) = B_m(p).$$

*Moreover, for  $m \leq k$  there exists  $y \in \bar{B}(p)$  such that  $q_m y < 0$ .*

The proof is based on the following lemma.

**Lemma 1** *Suppose that  $X \in \mathbb{R}^l$  is a polyhedral set and  $p \in {}^*\mathbb{R}^l$ . Then  $pX \geq 0$  implies  $p({}^*X) \geq 0$ .*

Note first that the conclusion of the lemma does not hold if  $X$  is not polyhedral. Take for instance  $X = \{x \in \mathbb{R}_+^2 \mid (x_2)^2 \geq x_1\}$  and  $p = (-\varepsilon, 1)$ ,  $\varepsilon \approx 0, \varepsilon > 0$ . Then, for each,  $x \in X$   $px = -\varepsilon x_1 + x_2 \geq 0$ . But once an element  $\tilde{x} = (\varepsilon/2, \varepsilon^2/4) \in {}^*X$  is taken,  $p\tilde{x} = -\varepsilon^2/2 + \varepsilon^2/4 < 0$ .

**Proof.** The set  $X$  consists of all vectors in  $\mathbb{R}^l$  that satisfy some system of linear inequalities:

$$X = \{x \in \mathbb{R}^l : c_\alpha x \leq b_\alpha, \alpha \in A\},$$

where  $c_\alpha \in \mathbb{R}^l \setminus \{0\}$ ,  $b_\alpha \in \mathbb{R}$ , and  $A$  is finite. By the Transfer principle,  ${}^*X$  consists of all elements in  ${}^*\mathbb{R}^l$  that satisfy the same system. Therefore,  ${}^*X$  has a finite number of faces and each face contains at least one standard element. Suppose that there exists  $\tilde{x} \in {}^*X$  such that  $p\tilde{x} < 0$ . A non-standard linear functional  $p$  is bounded from below (by any standard strictly negative number) on the internal closed set  ${}^*X$ , which means that

$$r = \min_{x \in {}^*X} px$$

exists and is strictly negative. A hyperplane  $H(p, r)$  is supporting to  ${}^*X$ , and  $F = H(p, r) \cap {}^*X$  is a face of a non-standard polyhedron  ${}^*X$ . But then  $py < 0$  for each  $y \in F$ , which contradicts to the fact that  $F$  contains at least one standard element.

□

**Corollary 1** *Suppose that  $X \in \mathbb{R}^l$  is star-shaped with respect to 0 and locally polyhedral at 0,  $0 \in X$  and  $p \in {}^*\mathbb{R}^l$ . Then  $pX \geq 0$  implies  $p({}^*X) \geq 0$ .*

**Proof.** Let  $P$  be a polyhedron obtained as an intersection of  $X$  with a sufficiently small closed cube, whose interior contains 0. Suppose that there exists  $\tilde{x} \in {}^*X$  such that  $p\tilde{x} < 0$ . Then one can find  $\tilde{x}' = \varepsilon\tilde{x} \in {}^*P$  such that  $p\tilde{x}' < 0$ , for some sufficiently small  $\varepsilon \in {}^*\mathbb{R}_{++}$ . But  $pP \geq 0$ , which by Lemma 1 implies  $p({}^*P) \geq 0$ , a contradiction.

□

**Proof of Proposition 5.** Let  $q^p = (q_1, \dots, q_k)$  for some  $k \leq l$ . If  $p = 0$  then  $q_1 = 0$ ,  $k = 1$  and  $\bar{B}(p) = X = B_1(p)$ . Suppose first that there exist a number  $m \in \{1, \dots, k\}$  and an element  $y \in X(q_1, \dots, q_{m-1})$  such that  $q_m y < 0$ . Moreover, assume that  $m$  is the smallest such a number, which guarantees that

$$q_j X(q_1, \dots, q_{j-1}) \geq 0, \quad j \in \{1, \dots, m-1\}. \quad (7)$$

Note that by construction  $y \in B_m(p)$ . If  $m = 1$  then  $\bar{B}(p) = B(q_1) = B_1(p)$  by virtue of Proposition 2. Assume  $m \in \{2, \dots, k\}$  and show that  $\bar{B}(p) = B_m(p)$ . First, we shall prove that  $\bar{B}(p) \subseteq B_m(p)$ . Take some arbitrary  $x \in \bar{B}(p)$  and suppose that  $x \notin B_m(p)$ . If so, then by the choice of  $m$  and the system of inequalities (7) there exists a number  $j \in \{1, \dots, m\}$  such that

$$q_j x > 0, \quad q_t x = 0, \quad t = 1, \dots, j-1. \quad (8)$$

Consider some arbitrary non-standard element  $\tilde{x} \in {}^*X$  such that  $\tilde{x} \approx x$ . Then  $q_j x > 0$  implies that  $q_j \tilde{x}$  is greater than some strictly positive real number.

Take now any standard  $x' \in X$  and consider the first non-zero element in the ordered set  $\{q_1 x', \dots, q_{j-1} x'\}$ . Since  $j \leq m$ , it follows again from (7) and the choice of  $m$  that such an element, if it exists, is strictly positive. Therefore, a non-standard linear functional  $\lambda_1 q_1 + \dots + \lambda_{j-1} q_{j-1}$  takes only positive values on  $X$ .

$$(\lambda_1 q_1 + \dots + \lambda_{j-1} q_{j-1})X \geq 0.$$

Then by Lemma 1

$$(\lambda_1 q_1 + \dots + \lambda_{j-1} q_{j-1}) {}^*X \geq 0. \quad (9)$$

Consider

$$\frac{1}{\lambda_j} p \tilde{x} = \frac{1}{\lambda_j} [\lambda_1 q_1 + \dots + \lambda_{j-1} q_{j-1}] \tilde{x} + q_j \tilde{x} + \frac{1}{\lambda_j} [\lambda_{j+1} q_{j+1} + \dots + \lambda_k q_k] \tilde{x}.$$

The first component of the sum in the right-hand side is positive (it vanishes if  $j = 1$ ), the second component exceeds 0 by a non-infinitesimal amount and the third component is infinitesimal. Therefore,  $(1/\lambda_j) p \tilde{x}$  is strictly positive, so that  $p \tilde{x} > 0$  for all  $\tilde{x} \in {}^*X$  such that  $\tilde{x} \approx x$ . This contradicts  $x \in \bar{B}(p)$ . We have shown that  $\bar{B}(p) \subseteq B_m(p)$ .

Let  $x \in B_m(p)$ ,  $y \in X(q_1, \dots, q_{m-1})$ ,  $q_m y < 0$ . Consider a sequence

$$x_n = \frac{1}{n} y + (1 - \frac{1}{n}) x.$$

By convexity,  $x_n \in {}^*X$  for any  $n \in {}^*\mathbb{N}$ . Moreover,  $p x_n < 0$  for all  $n \in \mathbb{N}$ . Show that  $p x_{\tilde{n}} < 0$  for some hyperfinite  $\tilde{n}$ . Suppose that the set

$$A = \{n \in {}^*\mathbb{N} : p x_n \geq 0\}$$

is non-empty. This set is internal as a definable subset of an internal set  ${}^*\mathbb{N}$  (see Davis (1977), Th. 1-8.1). Therefore it has a least element  $\nu \in {}^*\mathbb{N} \setminus \mathbb{N}$ . Take  $\tilde{n} = \nu - 1$ , then  $x_{\tilde{n}} \approx x$  and  $p x_{\tilde{n}} < 0$ , which proves that  $x \in \bar{B}(p)$ .

To complete the proof it suffices to show that

$$\forall m \in \{1, \dots, k\} \quad q_m X(q_1, \dots, q_{m-1}) \geq 0$$

implies  $\bar{B}(p) = B_{k+1}(p)$ . If  $x \in B_{k+1}(p)$  then  $p x = 0$ , so  $x \in \bar{B}(p)$ . The proof of the inclusion  $\bar{B}(p) \subseteq B_{k+1}(p)$  goes along the same lines as in the case

$m \leq k$ .

□

Next, we turn our attention to the characterization of non-standard dividend budget sets. For  $\gamma \in {}^*\mathbb{R}_{++}$  and  $p \in {}^*\mathbb{R}^l$  consider the set

$${}^*B(p, \gamma) = \{x \in {}^*X : px \leq \gamma\}$$

and denote by  $\bar{B}(p, \gamma)$  its standardization:

$$\bar{B}(p, \gamma) = st {}^*B(p, \gamma).$$

The following auxiliary lemma is useful. It says that small changes in prices and dividends do not alter a standardized dividend budget set.

**Lemma 2** *Let  $X$  be a closed convex set,  $0 \in X$ . Suppose that  $p, p' \in {}^*\mathbb{R}^l$  and that the non-standard numbers  $\gamma > 0, \gamma' > 0$  satisfy*

$$|p - p'| / \gamma \approx 0 \quad \text{and} \quad \gamma / \gamma' \approx 1,$$

*then  $\bar{B}(p, \gamma) = \bar{B}(p', \gamma')$ .*

**Proof.** Show first that  $\bar{B}(p, \gamma) = \bar{B}(p, \gamma')$ . To this end assume  $\gamma' > \gamma$  and show that the inclusion

$$\bar{B}(p, \gamma') \subseteq \bar{B}(p, \gamma) \tag{10}$$

holds. Let  $x \in \bar{B}(p, \gamma')$ . Then one can find  $\tilde{x} \approx x, \tilde{x} \in {}^*X$  such that  $p\tilde{x} \leq \gamma'$ . Suppose that  $p\tilde{x} > \gamma$  (otherwise there is nothing to prove), and consider  $y = (1 - \varepsilon)\tilde{x}$  where  $\varepsilon \approx 0$  satisfies  $\gamma' = (1 + \varepsilon)\gamma$ . It is clear that  $y \approx \tilde{x} \approx x$ ,  $y \in {}^*X$  by convexity, and

$$py = p\tilde{x} - \varepsilon p\tilde{x} \leq \gamma' - \varepsilon\gamma = \gamma.$$

Thus inclusion (10) follows.

Next we shall establish

$$\bar{B}(p, \gamma') = \bar{B}(p', \gamma').$$

Let  $p'' = p - p'$ . Since  $|p''| / \gamma' \approx 0$ , one can find  $\varepsilon \approx 0, \varepsilon > 0$  such that

$$|p''| / \varepsilon\gamma' \approx 0.$$

Then for every near-standard  $y \in {}^*X$

$$p'y - \varepsilon\gamma' \leq py \leq p'y + \varepsilon\gamma'.$$

Therefore  $py \leq \gamma'$  implies  $p'y \leq \gamma' + \varepsilon\gamma'$  and

$$\bar{B}(p, \gamma') \subseteq \bar{B}(p', \gamma' + \varepsilon\gamma') = \bar{B}(p', \gamma').$$

Similarly,  $\bar{B}(p', \gamma') \subseteq \bar{B}(p, \gamma')$ .

□

Using representation (2) for  $p \in {}^*\mathbb{R}^l$ , assign to each non-standard  $\gamma > 0$  its *infinitesimality level*, that is a number  $j = j(p, \gamma) \in \{1, \dots, k+1\}$  such that

$$j = \begin{cases} \min \{j \mid \gamma/\lambda_j \not\approx 0\}, & \text{if } \gamma/\lambda_k \not\approx 0, \\ k+1 & \text{otherwise.} \end{cases}$$

For  $j \leq k$ , denote by  $\mu = \mu(p, \gamma)$  a standard part of the ratio  $\gamma/\lambda_j$ :

$$\mu = {}^\circ(\gamma/\lambda_j).$$

Thus  $\mu$  is an element of  $\mathbb{R}_{++} \cup \{+\infty\}$ . Put  $\mu = +\infty$  if  $j = k+1$ .

The next statement gives a complete characterization of a non-standard dividend budget set for a polyhedral set  $X$ .

**Proposition 6** *Let  $X$  be polyhedral,  $p \in {}^*\mathbb{R}^l$ ,  $\gamma \in {}^*\mathbb{R}_{++}$ . Assume that  $0 \in X$ . Then one of the following alternatives is true:*

- (i)  $\bar{B}(p, \gamma) = X(q_1, \dots, q_{j-1})$  and  $\mu = +\infty$ ;
- (ii)  $\bar{B}(p, \gamma) = \{x \in X(q_1, \dots, q_{j-1}) \mid q_j x \leq \mu\}$  and  $\mu < +\infty$ ,
- (iii)  $\bar{B}(p, \gamma) = B_m(p)$  for some  $m < j$  and there exists  $y \in \bar{B}(p, \gamma)$  such that  $q_m y < 0$ .

Here  $q = (q_1, \dots, q_k)$  is a hierarchic price representing  $p$ ,  $j = j(p, \gamma)$  is the infinitesimality level of  $\gamma$ .

**Proof.** Consider an  $(l+1)$ -dimensional set  $X^* = X \times \{1\}$ , and a "budget" set

$$\{x \in {}^*X^* \mid p^*x \leq 0\}, \tag{11}$$

where the price vector  $p^* \in \mathbb{R}^{l+1}$  is given by

$$p^* = \begin{cases} (p', -\gamma), & \text{if } \gamma/\lambda_j \approx +\infty, \quad p' = p - \sum_{t \geq j} \lambda_t q_t, \\ (p', -\lambda_j \mu), & \text{if } \gamma/\lambda_j < +\infty, \quad p' = \sum_{t \leq j} \lambda_t q_t. \end{cases}$$

By construction  $|p - p'|/\gamma \approx 0$  and  $\lambda_j \mu/\gamma \approx 1$ . Since  $0 \in X$ , Lemma 2 implies that the projection of the standardization of the set defined in (11) onto the first  $l$  components of the set  $X^*$  coincides with  $\bar{B}(p, \gamma)$ . At the same time,

$$p^* = \sum_{t < j} \lambda_t q_t^* + \lambda_j^* q_j^*,$$



where  $q_t^* = (q_t, 0)$  if  $t < j$ ,

$$q_j^* = \begin{cases} (0, -1), & \text{if } \gamma/\lambda_j \approx +\infty, \\ (q_j, -\mu), & \text{if } \gamma/\lambda_j < +\infty \end{cases},$$

and

$$\lambda_j^* = \begin{cases} \gamma, & \text{if } \gamma/\lambda_j \approx +\infty, \\ \lambda_j, & \text{if } \gamma/\lambda_j < +\infty. \end{cases}$$

Since a hierarchic price  $\{q_t^*\}_{t \leq j}$  represents  $p^*$  (strictly speaking, the vector  $q_j^*$  should be normalized), Proposition 5 is applicable. The alternatives (i)–(iii) follow immediately. The case  $m < j$  corresponds to alternative (iii). If  $m = j$  and  $\gamma/\lambda_j < +\infty$ , then  $\mu < +\infty$ , and the budget restriction in the definition of  $B_m(p^*)$  has the form

$$\left((x, 1), (q_j, -\mu)\right) \leq 0 \Rightarrow q_j x \leq \mu,$$

so alternative (ii) follows. If  $m = j$  and  $\gamma/\lambda_j \approx +\infty$ , then alternative (i) is true. The case  $m = j + 1$  will not occur because the assumption  $0 \in X$  guarantees that  $0 \in \bar{B}(p, \gamma)$ .

□

This proposition makes it a mere formality to show that non-standard dividend equilibria coincide with Florig's hierarchic equilibria. One only needs to observe that non-standard prices and dividends on the one hand, and hierarchic prices and revenues on the other, provide an individual with the same budget opportunities. The reader can consult the appendix for a formal proof of this statement.

Propositions 5 and 6 allow us to set a useful relation between the standardization of the non-standard budget set  ${}^*B(p, \gamma)$  and its *standard part*, i.e. the set of all standard elements satisfying the constraint  $px \leq \gamma$ :

$${}^\circ B(p, \gamma) = \{x \in X : px \leq \gamma\}$$

**Proposition 7** *If  $X$  is a compact polyhedral set,  $0 \in X$ ,  $p \in {}^*\mathbb{R}^l$ , and  $\gamma \in {}^*\mathbb{R}_+$ , then*

$$\bar{B}(p, \gamma) = cl {}^\circ B(p, \gamma). \quad (12)$$

**Proof.** Since  ${}^\circ B(p, \gamma) \subseteq \bar{B}(p, \gamma)$  and since the latter set is closed, the claim is proved if for an arbitrary  $x \in \bar{B}(p, \gamma)$  we point out a standard sequence  $(x_\alpha)_{\alpha \in \mathbb{N}} \subset X$  converging to  $x$  such that

$$px_\alpha \leq \gamma \quad (13)$$

for every  $\alpha \in \mathbb{N}$ .

Suppose that  $\bar{B}(p, \gamma) = B_m(p)$  for some  $m < k + 1$ . Then there exists  $y \in \bar{B}(p, \gamma)$  such that  $q_m y < 0$ . We can choose for  $x_\alpha$  a convex combination

$$x_\alpha = (1/\alpha)y + (1 - 1/\alpha)x, \quad \alpha \in \mathbb{N}.$$

Actually, since

$$px_\alpha = (1/\alpha)\lambda_m q_m y + (1 - 1/\alpha)\lambda_m q_m x + \lambda_{m+1}[\dots] < 0,$$

where the value in the square brackets is near-standard (recall that  $x$  and  $y$  are near-standard), it is easy to see that (13) holds for every  $\gamma \in \mathbb{R}_+$ .

If the alternative (ii) of Proposition 6 is true, we can put

$$x_\alpha = (1 - 1/\alpha)x, \quad \alpha \in \mathbb{N}.$$

All other cases do not require special inspection because for every  $x \in \bar{B}(p, \gamma)$  we have  $px \leq \gamma$ , so it is possible to put  $x_\alpha = x$ .

□

In general, the relation (12) does not hold if  $X$  is not a polyhedron (see Marakulin (1988), page 24, for a counter-example).

## 4 Manifolds of hierarchic prices

As was mentioned before, a hierarchic price  $q = (q_1, \dots, q_k)$  is, by definition, an element of a Stiefel manifold  $V_{l,k}$ , which is a surface of dimension  $kl - k(k+1)/2$  (see Dubrovin et al. (1985)). This is intuitively clear, as  $q$  is parametrized by  $kl$  coordinates  $(q_{mt})_{m=1, \dots, k}^{t=1, \dots, l}$  related by  $k(k+1)/2$  equations:

$$\sum_{t=1, \dots, l} q_{mt} q_{m't} = \begin{cases} 1, & \text{if } m = m', \\ 0, & \text{if } m \neq m'. \end{cases}$$

The set of all possible hierarchic prices  $\Theta$  is then a union of  $k$  Stiefel manifolds and a vector zero:

$$\Theta = \left( \bigcup_{k=1}^l V_{l,k} \right) \cup \{0\},$$

where components  $\Theta$  of highest dimension are  $V_{l,l}$  and  $V_{l,l-1}$ ,

$$\dim V_{l,l} = \dim V_{l,l-1} = l(l-1)/2.$$

In this section we prove that if all consumption sets in an economy are polyhedral, then the set of all relevant hierarchic equilibrium prices  $\bar{\Theta}$  can be described as a finite union of manifolds whose dimension does not exceed

$l - 1$ . The term "relevant" means here that for each non-standard dividend equilibrium one can choose a hierarchic equilibrium price from  $\bar{\Theta}$ . Thus, the situation is similar to what we have in the purely standard case, where a price vector belongs to a sphere or an  $(l - 1)$ -simplex in  $\mathbb{R}^l$ . This is surprising, since it means that the introducing of non-standard or hierarchic prices does not change the dimension of the set of all possible equilibrium prices.

Besides, it means that if a non-standard equilibrium is described as a solution of a system of equations  $\psi(q) = 0$ , one can use an  $(l - 1)$ -dimensional set as a domain of correspondence  $\psi$ . This will be useful, when we begin to study generic properties of the set of non-standard equilibria.

**Theorem 1** *Suppose that  $X_i$  is a polyhedral set for every  $i \in N$ . Then there exists a set  $\bar{\Theta} \subset \Theta$  such that*

- (i) *for each non-standard dividend equilibrium  $\bar{x}$  there exists  $q \in \bar{\Theta}$  such that  $q$  is an equilibrium hierarchic price for  $\bar{x}$  (or, which is equivalent, such that  $q$  represents non-standard equilibrium prices  $p$  corresponding to the equilibrium  $\bar{x}$ ).*
- (ii)  *$\bar{\Theta}$  is a union of manifolds of dimension less than or equal to  $l - 1$ .*

**Proof.** We continue to use a convention  $w^i = 0$  for every  $i \in N$ . Suppose that  $\bar{x}$  is a non-standard dividend equilibrium and  $p \in {}^*\mathbb{R}^l$  and  $d \in {}^*\mathbb{R}_+^n$  are corresponding prices and dividends. It is possible to classify consumers according to the structure of their budget sets  $\bar{B}(p, d_i)$ . Proposition 6 implies that each consumer  $i$  faces (for some  $m = m(i) \in \{1, \dots, k + 1\}$ )  $m - 1$  budget constraints in the form of the equalities:

$$q_t x = 0, \quad t \in \{1, \dots, m - 1\},$$

and (possibly) one budget constraint in the form of the inequality:  $q_m x \leq 0$  or  $q_m x \leq \mu_j$ , where  $\mu_j = {}^\circ(d_j / \lambda_{j(p, d_i)})$ . Let  $N_m \subseteq N$  be the set of all agents for whom the last budget restriction involves the vector  $q_m$ . For all such agents we have

$$\bar{B}_i(p - \sum_{t > m} \lambda_t q_t, d_i) = \bar{B}_i(p, d_i).$$

Put  $i \in N_1$  if agent  $i$  faces no restrictions at all, that is if  $\bar{B}_i(p, d_i) = X_i$ . Thus, given non-standard prices  $p$  and dividends  $d_i$  we partition the set of agents  $N$  into  $k$  subsets  $N_1, \dots, N_k$ .

Consider an arbitrary agent  $i$ ;  $i \in N_m$  for some  $m$ . His budget set is a subset of the set

$$X_i(q_1, \dots, q_{m-1}) = \{x \in X_i \mid q_1 x = 0, \dots, q_{m-1} x = 0\}.$$

Each vector  $q_t$ ,  $t = 1, \dots, m-1$ , supports  $X_i(q_1, \dots, q_{t-1})$ , (though it does not have to support  $X_i$ ), which implies that the sets

$$X_i \supset X_i(q_1) \supset \dots \supset X_i(q_1, \dots, q_{m-1})$$

form a finite sequence of faces of  $X_i$  contained in each other. Denote the face  $X_i(q_1, \dots, q_{t-1})$  by  $F_i^t(p)$  (note that the superscript  $t$  specifies that there are  $t-1$  equalities).

Let us construct a set  $\Theta_p$  which contains a hierarchic price  $q^p = (q_1, \dots, q_k)$  representing  $p$ . It will be done in  $k$  steps, and  $\Theta_p$  will finally be obtained as a product of  $k$  spheres in some linear space.

Consider the set  $N_k$ . Without loss of generality it is not empty (otherwise one can throw away the last component of  $q^p$  and consider a new equilibrium price  $p' = p - \lambda_k q_k$ ). Let  $L_k$  be the linear hull of faces  $F_i^k$ ,  $i \in N_k$ ,

$$L_k = \text{span} \left( \bigcup_{i \in N_k} F_i^k(p) \right).$$

It is clear that the vector  $q_k$  must belong to this subspace (if necessary,  $q_k$  can be replaced by its projection on  $L_k$ ). Take a sphere in  $L_k$  as the last component of  $\Theta_p$ .

$$S_k = \{x \in L_k \mid \|x\| = 1\}.$$

Secondly, consider  $F_i^{k-1}(p)$  — superfaces of  $F_i^k(p)$  for  $i \in N_k$  — that is the sets

$$\{x \in X_i \mid q_t x = 0, \quad t \leq k-2\},$$

and faces  $F_i^{k-1}(p)$  for  $i \in N_{k-1}$ . Taking a linear hull of the union of the sets  $F_i^{k-1}(p)$  for all  $i$  from  $N_k$  and  $N_{k-1}$ , we obtain a linear space  $M_{k-1}$  that contains  $L_k$ . Denote by  $L_{k-1}$  an orthogonal complement to  $L_k$  in the space  $M_{k-1}$

$$L_{k-1} = M_{k-1} \cap (L_k)^\perp$$

and take a sphere in  $L_{k-1}$  as the next component of  $\Theta_p$

$$S_{k-1} = \{x \in L_{k-1} \mid \|x\| = 1\}.$$

(Note that since the vectors  $q_t$  are mutually orthonormal, the vector  $q_{k-1}$  always has a non-zero projection on  $L_{k-1}$ ).

The procedure described above is reiterated  $k-1$  times. As a result, a system of mutually orthogonal subspaces  $L_1, \dots, L_k$  is constructed, where

$$L_m = M_m \cap (L_{m+1})^\perp,$$

$$M_m = \text{span} \left( \bigcup_{i \in N_t, t \geq m} F_i^m(p) \right).$$

By construction,  $M_1 = \mathbb{R}^l$ ,  $M_1 = L_1 \oplus \cdots \oplus L_k$ . The set  $\Theta_p$  is defined as the product of spheres in  $L_m$ ,  $m = 1, \dots, k$ ,

$$\Theta_p = S_1 \times \cdots \times S_k,$$

so its dimension is equal to  $l - k$ .

Look now at the set

$$\Upsilon = \{\Theta_p \mid p \in {}^*\mathbb{R}^l \text{ is an equilibrium price of an economy } \mathcal{E}\}.$$

This set is finite because each  $X_i$  has only a finite number of faces. Finally, let

$$\bar{\Theta} = \bigcup_{\Theta_p \in \Upsilon} \Theta_p.$$

By construction, for every non-standard dividend equilibrium  $\bar{x}$  there exists a corresponding non-standard equilibrium price vector  $p$ , such that its hierarchic representation  $q^p$  belongs to  $\bar{\Theta}$ .

□

## 5 Optimality and Core equivalence

Marakulin (1990) has shown that each non-standard dividend equilibrium is weakly Pareto optimal and, conversely, that each weakly Pareto optimal allocation can be decentralized with an appropriate non-standard price vector. For an economy with polyhedral consumption sets, Proposition 7 implies that such an allocation (sometimes called a Pareto equilibrium) is a non-standard equilibrium after some appropriate redistribution of initial endowments, so the Second welfare theorem holds. Whether a Pareto equilibrium is always a non-standard equilibrium for an economy with consumption sets that are not necessarily polyhedral is still an open question.

Further in this section we generalize the core - dividend equilibria equivalence theorem (Konovalov (1998)) to the case where initial endowments of agents do not have to lie in the interior of their consumption set. We say that a coalition  $S \subseteq N$  *rejects* a feasible allocation  $\bar{x}$  if there exist a partition of  $S$  consisting of two subcoalitions  $S_1, S_2 \subseteq S$  and consumption bundles  $y^i \in X_i$ ,  $i \in S$ , such that

$$\sum_{i \in S} y^i = \sum_{i \in S_1} \bar{x}^i + \sum_{i \in S_2} w^i,$$

and  $y^i \in \mathcal{P}_i(\bar{x}^i)$ ,  $i \in S$ . Rejecting can be viewed as performed in three steps. First, each member  $i$  of  $S$  receives an offer  $\bar{x}^i$ . Second, a subcoalition  $S_1$  of agents who accept an offer is formed. At the third stage, "domestic" trade takes place. If it is possible to make all agents in  $S$  better off, then  $\bar{x}$  is said to be rejected by the coalition  $S$ .

Rejection is clearly a stronger notion than a traditional blocking. For this reason, the *rejective core* — the set of all feasible allocation that are not rejected by any coalition — is usually a proper subset of the core.

A fuzzy coalition  $\xi \in [0, 1]^N \setminus \{0\}$  rejects an allocation  $\bar{x} \in X$ , if there exist fuzzy coalitions  $t$  and  $s$  with  $t + s = \xi$  and consumption plans  $y^i \in X_i$ ,  $y^i \in \mathcal{P}_i(\bar{x}^i)$  for all  $i \in \text{supp } \xi$ , such that

$$\sum_{i \in N} \xi_i y^i = \sum_{i \in N} t_i \bar{x}^i + \sum_{i \in N} s_i w^i.$$

The set of all feasible allocations that cannot be rejected by any fuzzy coalition is called *the fuzzy rejective core* of an economy  $\mathcal{E}$  and is denoted by  $C_{fr}(\mathcal{E})$ .

It is easy to check that in Gale's (1976) example no dividend equilibria exist, while there is a whole continuum of the fuzzy rejective core allocations. However, it is possible to restore the core equivalence once dividend equilibria with non-standard prices are brought into consideration. Denote the set of non-standard dividend equilibria of an economy  $\mathcal{E}$  by  $W_{nd}(\mathcal{E})$ .

**Theorem 2** *Assume that all consumption sets  $X_i$  are polyhedral, and that the sets  $\mathcal{P}_i(x^i)$  are open, convex and do not contain  $x^i$  for all  $x^i \in X_i, i \in N$ . Then the fuzzy rejective core of the economy  $\mathcal{E}$  coincides with the set of non-standard dividend equilibria:*

$$C_{fr}(\mathcal{E}) = W_{ns}(\mathcal{E}).$$

**Proof.** First, we show that

$$C_{fr}(\mathcal{E}) \subseteq W_{ns}(\mathcal{E}).$$

Let  $\bar{x} = (\bar{x}^i)_{i \in N} \in C_{fr}(\mathcal{E})$  and consider the sets

$$\mathcal{G}^i(\bar{x}^i) = \{y^i - \bar{x}^i | y^i \in \mathcal{P}_i(\bar{x}^i)\}, \quad i \in N,$$

and

$$\mathcal{G}^i(w^i) = \{y^i - w^i | y^i \in \mathcal{P}_i(\bar{x}^i)\}, \quad i \in N.$$

Denote the convex hull of the union of these sets by  $G$

$$G = \text{conv} \bigcup_{i \in N} (\mathcal{G}^i(\bar{x}^i) \cup \mathcal{G}^i(w^i))$$

and show that  $G$  does not contain zero. Suppose it does. Then, by convexity of  $\mathcal{G}^i(\bar{x}^i)$  and  $\mathcal{G}^i(w^i)$ , there exist  $t = (t_i)_{i \in N}$ ,  $s = (s_i)_{i \in N}$ ,  $y, z \in X$  such that

$$\begin{aligned} \sum_{i \in N} t_i + \sum_{i \in N} s_i &= 1, \quad t_i, s_i \geq 0, \quad i \in N, \\ \sum_{i \in N} t_i y^i + \sum_{i \in N} s_i z^i &= \sum_{i \in N} t_i \bar{x}^i + \sum_{i \in N} s_i w^i, \end{aligned} \quad (14)$$

where  $y^i \in \mathcal{P}_i(\bar{x}^i)$  if  $i \in \text{supp } t$ , and  $z^i \in \mathcal{P}_i(\bar{x}^i)$  if  $i \in \text{supp } s$ . This implies that  $\bar{x}$  is rejected by a fuzzy coalition  $t + s$ , a contradiction. Therefore, a zero point does not belong to the convex set  $G$ . By Theorem 1.2 in Marakulin (1988) we can strictly separate 0 from  $G$  with a non-standard hyperplane. In other words, there exists  $p \in {}^*\mathbb{R}^l$  such that for every  $y^i \in \mathcal{P}_i(\bar{x}^i)$ ,  $i \in N$ , the following conditions hold simultaneously

$$py^i > pw^i, \quad (15)$$

and

$$py^i > p\bar{x}^i. \quad (16)$$

Define the components of the vector  $d \in {}^*\mathbb{R}_+^n$  by

$$d_i = \max \{0, p\bar{x}^i - pw^i\}, \quad i \in N. \quad (17)$$

Then  $p\bar{x}^i \leq pw^i + d_i$ ,  $i \in N$ , which implies attainability of  $\bar{x}$ . To prove the validity of the required inclusion we have to show the individual rationality of  $\bar{x}$ :

$$\bar{B}_i(p, d_i) \cap \mathcal{P}_i(\bar{x}^i) = \emptyset. \quad (18)$$

What we do have so far is

$$\{x \in {}^*X_i : px \leq pw^i + d_i\} \cap \mathcal{P}_i(\bar{x}^i) = \emptyset, \quad i \in N.$$

But then (18) is a consequence of Proposition 7 and openness of the sets  $\mathcal{P}_i(\bar{x}^i)$ ,  $i \in N$ .

To prove the converse inclusion  $W_{ns}(\mathcal{E}) \subseteq C_{fr}(\mathcal{E})$ , let  $\bar{x} \in W_{ns}(\mathcal{E})$  and assume that  $p \in {}^*\mathbb{R}^l$  and  $d \in {}^*\mathbb{R}_+^n$  are corresponding non-standard equilibrium prices and dividends. Suppose that there exists a fuzzy coalition  $\xi = (\xi_i)_{i \in N}$  that blocks  $\bar{x}$ . Then, there exist  $y^i \in \mathcal{P}_i(\bar{x}^i)$ ,  $t_i, s_i \in \mathbb{R}_+$ ,  $i \in \text{supp } \xi$  such that

$$\sum_{i \in N} t_i y^i + \sum_{i \in N} s_i y^i = \sum_{i \in N} t_i \bar{x}^i + \sum_{i \in N} s_i w^i, \quad (19)$$

where  $t_i + s_i = \xi_i$ . By individual rationality,

$$y^i \notin \bar{B}_i(p, d_i), \quad i \in \text{supp } \xi, \quad (20)$$

which implies that

$$py^i > pw^i, \quad i \in \text{supp } \xi.$$

Fix  $i \in \text{supp } \xi$  and consider a hierarchic representation  $(q_1, \dots, q_k)$  of prices  $p$ . For every  $y^i$  find a number  $h = h(i) \in \{1, \dots, k\}$  such that

$$q_h y^i > q_h w^i, \quad q_r y^i = q_r w^i, \quad r < h.$$

We claim that

$$e_h y^i > e_h \bar{x}^i, \quad i \in \text{supp } \xi. \quad (21)$$

To prove it, use the characterization of the budget set  $\bar{B}_i(p, d_i)$  given by Proposition 6. Since  $y^i \notin \bar{B}_i(p, d_i)$ , there exists  $m = m(i) \in \{1, \dots, k\}$  such that at least one of the conditions

$$\begin{aligned} q_1 y^i &= q_1 w^i, \\ &\vdots \\ q_{m-1} y^i &= q_{m-1} w^i, \end{aligned}$$

$$q_m y^i \leq q_m w^i + \mu_i \vee q_m y^i \leq q_m w^i,$$

where  $\mu_i = {}^\circ(d_i / \lambda_{j(p, d_i)})$ , is violated, so it must be  $h \leq m$ . Taking into account  $\bar{x}^i \in \bar{B}_i(p, d_i)$  one obtains  $q_h y^i > q_h \bar{x}^i$ . Moreover, it is easy to see that  $q_r \bar{x}^i = q_r w^i = q_r y^i$  for all  $r < h$ . Define

$$\bar{h} = \min_{i \in \text{supp } \xi} h(i).$$

Then for every  $i \in \text{supp } \xi$

$$q_{\bar{h}} y^i \geq q_{\bar{h}} w^i, \quad (22)$$

$$q_{\bar{h}} y^i \geq q_{\bar{h}} \bar{x}^i, \quad (23)$$

and there necessarily exists  $i \in \text{supp } \xi$  such that both inequalities are strict. Multiplying (22) and (23) by  $s_i$  and  $t_i$  respectively and summing over all  $i$  gets

$$(q_{\bar{h}}, \sum_{i \in N} t_i y^i) + (q_{\bar{h}}, \sum_{i \in N} s_i y^i) > (q_{\bar{h}}, \sum_{i \in N} t_i \bar{x}^i) + (q_{\bar{h}}, \sum_{i \in N} s_i w^i),$$

which contradicts (19). Consequently, the inclusion  $W_{ns}(\mathcal{E}) \subseteq C_{fr}(\mathcal{E})$  is true and this completes the proof of the theorem.

□

As we already know, the set of non-standard dividend equilibria is equal to the set of hierarchic equilibria for an economy with polyhedral consumption sets. Therefore for such an economy the set of hierarchic equilibria coincide with the fuzzy rejective core. Thus we have obtained a core equivalence theorem by Florig (1999) as a corollary of our results.



## 6 Existence of non-standard equilibria

Starting with this section, we assume that preferences of agents are given by utility functions defined over the set  $X = \prod_{i \in N} X_i$

$$u_i : \prod_{i \in N} X_i \rightarrow \mathbb{R}, \quad i \in N.$$

However, all results of the section can easily be generalized to the case of non-complete and non-transitive preferences. We introduce a new concept of non-standard dividend equilibrium similar to the one given in Marakulin (1990) but with a specified system of dividends. If the survival assumption is satisfied in the model, then this concept boils down to a notion of equilibrium with individual slacks proposed by Kajii (1996).

Fix a *standard* strictly positive vector  $\delta \in \mathbb{R}_{++}^n$ . For a given non-standard number  $\varepsilon \in {}^*\mathbb{R}_+$  consider the dividend budget sets

$$\bar{B}_i(p, \varepsilon \delta_i) = st \{x \in X_i \mid px \leq pw^i + \varepsilon \delta_i\}, \quad i \in N.$$

**Definition 3** An allocation  $\bar{x}$  is called a  $\delta$ -*equilibrium* of an economy  $\mathcal{E}$ , if there exist  $\varepsilon \in {}^*\mathbb{R}_+$  and  $p \in {}^*Q$  such that the following conditions hold:

(i) attainability:

$$\bar{x}^i \in \bar{B}_i(p, \varepsilon \delta_i), \quad i \in N,$$

(ii) individual rationality:

$$u_i(\bar{x}) = \max_{x^i \in \bar{B}_i(p, \varepsilon \delta_i)} u_i(\bar{x}|x^i), \quad i \in N,$$

(iii) market clearing:

$$\sum_{i \in N} \bar{x}^i = \sum_{i \in N} w^i.$$

The specifics of  $\delta$ -equilibria are twofold. First, the ratio of individual dividends is assumed to be given a priori and fixed, as in the case of Kajii's equilibria with individual slacks, or, for instance, Mas-Colell's (1992) equilibria with slack, where the uniform dividend scheme was applied. Second, dividends of all consumers have the same "order of smallness"  $\varepsilon$ . Therefore, income is redistributed at most at one infinitesimality level, which may not generally be the case for non-standard dividend equilibria. Interpretation of the components  $\delta_i$  depends on the further specification of the model. For instance, they may represent initial stocks of coupons or paper money or express market shares of individuals.

We show first that  $\delta$ -equilibria exist even if the survival condition or any of its analogues is not satisfied. Note that the existence of non-standard dividend equilibria unlike that of hierarchic equilibria (see Florig (1998b)) does not require any conditions on consumption sets aside from convexity and compactness. Moreover, the compactness assumption can always be relaxed and substituted for closedness and boundedness from below.

**Theorem 3** *Let  $Q = \{p \in \mathbb{R}^l : \|p\| \leq 1\}$ . Assume that the set  $X_i$  is convex and compact, and the utility function  $u_i$  is continuous in  $x$  and strictly quasi-concave in  $x^i$  for every  $i \in N$ . Then for each  $\delta \in \mathbb{R}_{++}^n$  a  $\delta$ -equilibrium exists.*

The idea of the proof is to change slightly (in fact, by an infinitesimal value) agents' income functions  $pw^i$  to make them meet the strong survival assumption, and apply a non-standard translation of the standard existence theorem. Kajii's (1996) theorem is appropriate for this purpose. An equilibrium allocation is found as a standard part of a non-standard equilibrium allocation obtained in such a way.

**Proof.** Let us take an arbitrary standard strictly positive  $\gamma > 0$  and extend the economy with a new consumer 0 having a consumption set

$$X_0 = \{x \in \mathbb{R}^l \mid \|x\| \leq \gamma\},$$

initial endowments  $w^0 = 0$ , and utility function  $u_0 = 0$ . Revise income functions of individuals: instead of  $pw^i$ , consider functions

$$\begin{aligned} \alpha_0(p) &= -\gamma\|p\|/2, \\ \alpha_i(p) &= pw^i + \frac{\gamma\delta_i\|p\|}{2\sum_{j \in N} \delta_j}, \quad i \in N, \end{aligned}$$

as new incomes. At each non-zero prices some amount of value is taken from consumer 0 and redistributed among other agents in proportions given by  $\delta$ . Put  $N_0 = N \cup \{0\}$ . We have obtained an exchange economy with  $n + 1$  consumers and generalized income functions

$$\mathcal{E}^\gamma = \langle N_0, \{X_i, u_i, \alpha_i, w^i\}_{i \in N_0}, Q \rangle.$$

The functions  $\alpha_i$  are continuous, satisfy Walras' law:

$$\sum_{i=0}^n \alpha_i(p) = -\frac{\gamma\|p\|}{2} + p \sum_{i \in N} w^i + \frac{\gamma\|p\| \sum_{i \in N} \delta_i}{2\sum_{j \in N} \delta_j} = p \sum_{i \in N} w^i,$$

and satisfy the strong survival assumption: for each non-zero  $p \in Q$

$$\alpha_0(p) = -\gamma\|p\|/2 > -\gamma\|p\| = \inf pX_0,$$

$$\alpha_i(p) > pw^i \geq \inf pX_i, \quad i \in N.$$

By Corollary 1 in Kajii (1996), for each  $\gamma > 0$  there exists some  $\varepsilon_\gamma \geq 0$ , a pair  $(x_\gamma^0, x_\gamma) \in X_0 \times X$ , and a price vector  $p_\gamma \in Q$  such that each  $x_\gamma^i$  maximizes  $u_i$  on the set

$$B_i(p_\gamma, \alpha_i, \varepsilon_\gamma \delta_i) = \{x \in X_i \mid p_\gamma x \leq \alpha_i(p_\gamma) + \varepsilon_\gamma \delta_i\},$$

and total demand is equal to total supply

$$x_\gamma^0 + \sum_{i \in N} x_\gamma^i = \sum_{i \in N} w^i.$$

Choose any strictly positive infinitesimal  $\gamma \in {}^*\mathbb{R}_{++}$ . By the Transfer Principle, we can find  $\tilde{\varepsilon} \in {}^*\mathbb{R}_+$ , an allocation  $(\tilde{x}^0, \tilde{x}) \in {}^*X_0 \times {}^*X$  and prices  $\tilde{p} \in {}^*Q$  such that each  $\tilde{x}^i$  maximizes  $u_i$  on the set

$${}^*B_i(\tilde{p}, \alpha_i, \tilde{\varepsilon} \delta_i) = \{x \in {}^*X_i \mid \tilde{p}x \leq \alpha_i(\tilde{p}) + \tilde{\varepsilon} \delta_i\},$$

and

$$\tilde{x}^0 + \sum_{i \in N} \tilde{x}^i = \sum_{i \in N} w^i. \quad (24)$$

Since  $\tilde{x}$  belongs to  ${}^*X$  and  $X$  is a compact set,  $\tilde{x}$  is near-standard. Let  $\bar{x} = st \tilde{x}$  be a standard part of  $\tilde{x}$ . We shall prove that  $\bar{x}$  is a  $\delta$ -equilibrium provided prices are  $\tilde{p}$  and

$$\varepsilon = \tilde{\varepsilon} + \frac{\gamma \|\tilde{p}\|}{2 \sum_{i \in N} \delta_i}.$$

Taking standard parts from both sides of (24), one easily gets market clearing:

$$\sum_{i \in N} \bar{x}_i = \sum_{i \in N} w^i.$$

Moreover, for each  $i$

$$\bar{x}^i \in \bar{B}_i(\tilde{p}, \varepsilon \delta_i) = st \{x \in {}^*X_i \mid \tilde{p}x \leq \tilde{p}w^i + \varepsilon \delta_i\}.$$

To complete the proof, we need to show that  $\bar{x}^i$  maximizes  $u_i$  on the set  $\bar{B}_i(\tilde{p}, \varepsilon \delta_i)$  given that  $\tilde{x}^i$  is a utility maximum on the set

$${}^*B_i(\tilde{p}, \varepsilon \delta_i) = \{x \in {}^*X_i \mid \tilde{p}x \leq \tilde{p}w^i + \varepsilon \delta_i\}.$$

Suppose that for some  $i \in N$  it does not hold. Then there exists a standard  $y \in \bar{B}_i(\tilde{p}, \varepsilon \delta_i)$  such that  $u_i(y) > u_i(\bar{x}^i)$ . By the definition of the set  $\bar{B}_i(\tilde{p}, \varepsilon \delta_i)$

there should exist  $y' \in {}^*B(\tilde{p}, \varepsilon\delta_i)$  such that  $y' \approx y$ . By continuity of the utility function  $u_i$ ,

$$u_i(y') > u_i(x')$$

for any  $x' \approx \tilde{x}^i$ . Therefore,  $u_i(y') > u_i(\tilde{x}^i)$ , a contradiction with  $\tilde{x}^i$  being the best element on the set  ${}^*B_i(\tilde{p}, \varepsilon\delta_i)$ .

□

Obviously, existence of  $\delta$ -equilibria implies existence of non-standard dividend equilibria. Thus we have the theorem of Marakulin (1990) as a corollary.

## 7 Finiteness of non-standard equilibria

Gerard Debreu (1970), one of the founders of equilibrium analysis, was the first to establish the finiteness of equilibria for "almost all" exchange economies. His approach was based on a variation of initial endowments while all other parameters of the model were fixed. Using Sard's theorem applied to the aggregate excess demand function, Debreu obtained finiteness of equilibria for an open set containing almost all – in the sense of Lebesgue measure – allocations of initial resources. In subsequent contributions to the issue, Smale (1974), Dubey (1980) and others used also variations of utility functions, which required the use of Thom's theorems of openness and density of transversal intersections.

In the present paper we follow the latter approach, i.e., only utility functions vary, while initial endowments are kept fixed. This is done because of the fact that for almost all initial allocations of resources, the survival condition is satisfied, in which case non-standard equilibria coincide with usual Walrasian equilibria. Their local uniqueness follow then by the Debreu (1970) theorem.

We start this section with an example which illustrates that the system of dividends has to be specific for the number of non-standard equilibria to be finite. Let  $X_1 = X_2 = X_3 = \{(x_1, x_2) : 0 \leq x_j \leq 10, j = 1, 2\}$ ,  $Q = \{p \in \mathbb{R}^l : \|p\| \leq 2\}$ ,  $w^1 = (2, 1)$ ,  $w^2 = w^3 = (2, 0)$ ,  $u_1(x) = 5 - (x_1 - 1)^2 - (x_2 - 2)^2$ ,  $u_2(x) = u_3(x) = x_1$ . Allocations  $\bar{x}^1 = (1, 1)$ ,  $\bar{x}^2 = (2 + \lambda, 0)$ ,  $\bar{x}^3 = (3 - \lambda, 0)$ ,  $0 \leq \lambda \leq 1$  constitute a continuum of non-standard dividend equilibria for  $p = (\varepsilon, 1)$ ,  $\varepsilon \approx 0$ ,  $\varepsilon > 0$ , and dividends  $d = (0, \lambda\varepsilon, (1 - \lambda)\varepsilon)$ . This example is robust against sufficiently small perturbations of utility functions. Observe that if variations of initial endowments are considered, then the number of non-standard equilibria is generically finite. Indeed, the survival assumption ( $w^i \gg 0$ ,  $i = 1, 2, 3$ ) is satisfied for almost all perturbations of initial endowments, in which case non-standard dividend equilibria coincide with ordinary Walrasian equilibria (see Proposition 2).

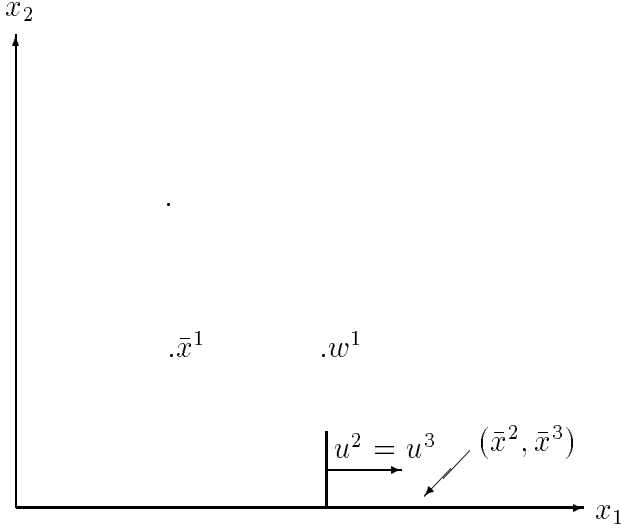


Figure 2: Continuum of non-standard dividend equilibria.

Furthermore, we will focus our attention on a specific class of  $\delta$ -equilibria, for which there exists at least one agent who consumes an element of the interior of his consumption set.

**Definition 4** A  $\delta$ -equilibrium  $\bar{x}$  is *proper* if there exists  $i_0 \in N$  such that  $\bar{x}^{i_0} \in \text{int } X_{i_0}$ .

In the rest of the paper, we hold  $X^i$  and  $w^i$  fixed for every  $i \in N$ . The finiteness result is established under the following **assumptions**:

- A1.** For all  $i \in N$  the set  $X_i$  is a bounded from below polyhedron with a non-empty interior.
- A2.** Utility functions  $u^i$  are defined and twice differentiable on an open neighbourhood  $\tilde{X}$  of the set  $X$ .

Denote by  $U$  the linear space  $C^2(\tilde{X}, \mathbb{R}^n)$  and endow it with the standard topology of  $C^2$  uniform convergence on compacts: if  $\{f_t\}_{t=1}^\infty \subset C^2(\tilde{X}, \mathbb{R}^n)$ , then  $f_t \rightarrow f_0 \in C^2(\tilde{X}, \mathbb{R}^n)$  if and only if  $f_t|_K \rightarrow f_0$  if  $t \rightarrow \infty$  in the norm  $\|\cdot\|_{C^2}$  of the vector space  $C^2(K, \mathbb{R}^n)$  for every compact set  $K \subset \tilde{X}$ . The norm  $\|\cdot\|_{C^2}$  is defined by

$$\begin{aligned} \|g\|_{C^2(K, \mathbb{R}^n)} &= \max\{\|g_i\|_{C(K, \mathbb{R}^n)}, \quad \left\|\frac{\partial g_i}{\partial x_j}\right\|_{C(K, \mathbb{R}^n)}, \\ &\quad \left\|\frac{\partial^2 g_i}{\partial x_j \partial x_s}\right\|_{C(K, \mathbb{R}^n)}, \quad i \in N, \quad j, s \in N \times L\}, \end{aligned}$$

where

$$\|g\|_{C(K, \mathbb{R}^n)} = \max\{|g(x)| : x \in K\}.$$

Thus we can think of an economy as of given by an element  $u$  of the set  $U$ .

Recall that a subset of a topological space is called *residual* if it is the countable intersection of open dense sets. In particular, the Baire Category Theorem asserts that a residual subset of a complete metric space is dense. The main result of this section is the following theorem.

**Theorem 4** *For any strictly positive vector  $\delta$ , there exists a residual (of the second category and hence dense) set  $G \subseteq U$  such that for each  $u \in G$  the set of proper  $\delta$ -equilibria is finite.*

The proof is rather cumbersome and will be organized into a number of steps. We give now a brief guide how the finiteness of  $\delta$ -equilibria will be obtained. First we consider the set of equilibrium hierarchic prices  $\Theta_F$  that correspond to  $\delta$ -equilibria from the relative interior of an arbitrary face  $F$  of the polyhedron  $X$ . After that, we construct a mapping  $\Psi_u$ ,  $u \in U$ , which is defined on  $ri F \times \Theta_F$  and takes its values in some finite-dimensional space. This mapping characterizes  $\delta$ -equilibria from  $ri F$ . Then, in the range of  $\Psi_u$ , we find a manifold  $\Delta_F$  such that  $\Psi_u^{-1}(\Delta_F)$  contains all  $\delta$ -equilibria from  $ri F$ . Thus, to establish finiteness of  $\delta$ -equilibria, it is sufficient to show that the manifold  $\Psi_u^{-1}(\Delta_F)$  has dimension zero (is discrete). We show that it is indeed so if  $\Psi_u$  is transversal to  $\Delta_F$ . Finally, it follows by Thom's theorems of density and openness of transversal sections that the mapping  $\Psi_u$  is transversal to  $\Delta_F$  for a residual set of economies  $u \in U$ .

In our argument we heavily rely on Proposition 4 that gives a representation of a non-standard price vector by a hierarchic price and Proposition 6 that provides us with the characterization of non-standard dividend budget sets. When using Proposition 6 to characterize  $\delta$ -equilibria, one should remember that all dividend terms  $\varepsilon\delta_i$  have the same infinitesimality level  $j(p, \varepsilon\delta_i) = j(p, \varepsilon)$ . Moreover, it follows from Lemma 2 that all components of an equilibrium hierarchic price which have a number higher than  $j$  do not matter. Hence, we can assume without loss of generality that for any hierarchic representation  $q = \{q_1, \dots, q_k\}$  of non-standard equilibrium prices  $p$

$$k = j(p, \varepsilon) \quad \text{and} \quad \varepsilon/\lambda_k < +\infty.$$

The case  $\varepsilon/\lambda_k \approx +\infty$  is not interesting, since it can be reduced to the prices  $p' = \sum_{j < k} \lambda_j q_j$  for which  $\varepsilon/\lambda_{j-1} \approx 0$ .

Proposition 6 implies that each agent  $i$  faces  $m - 1$  budget restrictions in the form of equalities:

$$q_t x = q_t w_i, \quad t < m, \quad x \in X_i, \quad (25)$$

and one restriction in the form of inequality

$$q_m x \leq q_m w^i, \quad x \in X_i,$$

for some natural number  $m < k$ . Moreover, there always exists  $y \in X_i$  for which the last inequality is strict. For  $m = k$  the last restriction transforms to

$$q_k x \leq q_k w^i + \delta_i \mu, \quad \mu = {}^\circ(\varepsilon/\lambda_k) \quad (26)$$

and can be realized as an equality if  $\mu = 0$ . Equations (25) determine a face  $F_i^m(p)$  of the polyhedron  $X_i$  such that  $w^i \in F_i^m(p)$ . In other words,

$$\bar{B}_i(p, \varepsilon \delta_i) = \{x \in F_i^m(p) \mid q_m x \leq q_m w^i\} \quad (27)$$

for  $m < k$ , and

$$\bar{B}_i(p, \varepsilon \delta_i) = \{x \in F_i^k(p) \mid q_k x \leq q_k w^i + \delta_i \mu\} \quad (28)$$

otherwise.

From now on we again use a convention  $w^i = 0, i \in N$  in order to facilitate notations. Consider a partition  $\mathcal{N}(p) = \{N_1, \dots, N_k\}$  of the set  $N$  such that each set  $N_m$  contains the agents, whose last budget restriction involves  $q_m$ .

Fix some face  $F \subset X$  and consider a  $\delta$ -equilibrium  $x$  that belongs to  $ri F$  — the relative interior of  $F$ . Let  $p$  be the corresponding non-standard equilibrium price vector. It is easy to see that

$$ri F_i \subset F_i^m(p), \quad i \in N,$$

where  $F_i$  are faces of  $X_i$  that compose the face  $F$  :

$$F = \prod_N F_i.$$

Faces  $F_i^m(p) \subset X^i$  are determined by prices  $p$  and equalities (25) for the appropriate  $m \leq k$ . The definition of a  $\delta$ -equilibrium implies that each  $x^i$  maximizes utility  $u_i(\cdot)$  on the set

$$\{x \in ri F_i \mid q_m x \leq 0\}$$

if  $m < k, i \in N_m$  and on the set

$$\{x \in ri F_i \mid q_k x \leq \delta_i \mu\}$$

if  $i \in N_k$ .

Let  $\Theta_F$  be the set that contains all hierarchic equilibrium prices that correspond to  $\delta$ -equilibria from  $F$ . The construction used in the proof of Theorem 1 implies that this set can be represented as a finite union of manifolds  $\Theta_F^\xi$  such that for each  $\Theta_F^\xi$  there exists  $k \in \{1, \dots, l\}$  such that

$$\Theta_F^\xi = \prod_{m=1}^k (L_m^\xi \setminus \{0\}),$$

where  $L_1^\xi, \dots, L_k^\xi$  are mutually orthogonal subspaces of  $\mathbb{R}^l$ . From now on, we fix an arbitrary element  $\Theta$  of this finite union and denote by  $\Theta^m$  its components  $L_m \setminus \{0\}$ ,  $m = 1, \dots, k$ .

$$\Theta = \Theta^1 \times \dots \times \Theta^k.$$

We proceed with introducing a mapping  $\Psi_u$  that characterizes  $\delta$ -equilibria from the face  $F$ . This mapping will be constructed as a product of the following correspondences.

Mappings  $\Psi_i^u : \tilde{X} \rightarrow \mathbb{R}^{L \times \{i\}}$ ,  $i \in N$  are defined by fragments of the gradient vectors of agents' utility functions related to their own consumption:

$$(\Psi_i^u)_j(x) = \frac{\partial u^i}{\partial x_j^i}(x), \quad j \in L.$$

Mappings  $\Psi_i^F$ ,  $i \in N$  reflect a condition that a  $\delta$ -equilibrium allocation belongs to the face  $F = \prod_{i \in N} F_i$ . Remind that we consider only  $x^i \in ri F_i$ . We replace here this condition with the milder requirement  $x^i \in span F_i$ . Choose vectors  $c_t^i \in \mathbb{R}^l$  such that  $x^i \in span F_i$  iff  $x^i$  is a solution of a system of  $t(i)$  linear equations

$$c_t^i x^i = 0, \quad t \in T_i, \quad T_i = \{1, \dots, t(i)\}, \quad (29)$$

and suppose that all rows of a matrix

$$C_i = \begin{pmatrix} c_1^i \\ \vdots \\ c_{t(i)}^i \end{pmatrix}$$

are linearly independent for each  $i \in N$ . A mapping  $\Psi_i^F : \tilde{X} \rightarrow \mathbb{R}^{T_i}$  is defined by

$$\Psi_i^F(x) = C_i x^i.$$



Thus, a condition  $\Psi_i^F(x) = 0$  is necessary for  $x^i$  to belong to  $ri\ F_i$ .

Let  $N_1, \dots, N_k$  be an arbitrary partition of the set  $N$ . A mapping  $\Psi^m : \tilde{X} \times \Theta^m \rightarrow \mathbb{R}^{N_m \setminus \{i_0\}}$  responds to the budget restriction  $q_m x^i = 0$  for  $i \in N_m$  :

$$(\Psi^m(x, q_m))_i = q_m x^i, \quad i \in N_m \setminus \{i_0\}, \quad m = 1, \dots, k.$$

For technical reasons, the budget restriction for agent  $i_0$  is removed. This restriction follows from those of other individuals and feasibility of an equilibrium allocation.

A mapping  $\Psi^{mc} : \tilde{X} \rightarrow \mathbb{R}^{L \times \{n+1\}}$  represents an equilibrium market clearing condition:

$$(\Psi^{mc}(x))_j = \sum_{i \in N} x_j^i, \quad j \in L.$$

Finally, we need the identity mapping  $\Psi^q : \Theta \rightarrow \Theta$ ,

$$\Psi^q(q_1, \dots, q_k) = (q_1, \dots, q_k).$$

A mapping  $\Psi_u$  is defined as a product of the correspondences described above:

$$\Psi_u = \prod_{i \in N} \Psi_i^u \times \Psi^{mc} \times \prod_{i \in N} \Psi_i^F \times \prod_{m=1}^k \Psi^m \times \Psi^q.$$

This mapping has a domain  $Z = \tilde{X} \times \Theta$  and takes its values in a finite-dimensional space  $\mathbb{R}^T \times \Theta$ , where

$$T = \left( \bigcup_{i=1}^{n+1} L \times \{i\} \right) \cup (N \setminus \{i_0\}) \cup \left( \bigcup_{i \in N} T_i \right).$$

A mapping  $\Psi : U \times Z \rightarrow \mathbb{R}^T \times S$  is defined by

$$\Psi(u, z) = \Psi_u(z).$$

At the next step, we describe a submanifold  $\Delta_F$  such that for each proper  $\delta$ -equilibrium  $(x, q_1, \dots, q_k)$  its value  $\Psi_u(x, q_1, \dots, q_k)$  belongs to  $\Delta_F$  for some choice of parameters  $i_0, \Theta, N_1, \dots, N_k$ .

Divide the set  $N_m$  into two subsets  $N'_m$  and  $N''_m$  assuming that  $N'_m$  consists only of those agents for whom the last budget restriction is binding. The other set should contain then only those agents from  $N_m$  who are locally satiated on the face  $F_i$ . The necessary first-order conditions of a local extremum for the agents of the first type can be formulated as follows: there exist  $y \in \mathbb{R}^{T_i}$  and  $\lambda^m \in \mathbb{R}_{++}$  such that

$$\Psi_i^u(x) = \frac{\partial u_i}{\partial x^i}(x) = \lambda^m q_m + y C_i.$$

For agents of the second type those conditions have a simpler form: there exists  $y \in \mathbb{R}^{T_i}$  such that

$$\Psi_i^u(x) = \frac{\partial u_i}{\partial x^i}(x) = y C_i.$$

When defining a manifold  $\Delta_F$ , we take into consideration that  $\Psi^m(x, q)$  is equal to zero for agents from  $N'_m$  whenever  $m < k$ , or to  $\mu \delta_i$  if  $m = k$ ; and corresponds to a free variable for  $i \in N''_m$ ,  $m = 1, \dots, k$ .

Define a submanifold  $\Delta_F \subset \mathbb{R}^T \times \Theta$  as follows:

$$\Delta_F \stackrel{def}{=} \left\{ (\nu_1, \dots, \nu_n, \beta_1, \dots, \beta_k, \sigma, q, \varphi_1, \dots, \varphi_n) \in \mathbb{R}^T \times \Theta \mid \right.$$

$$\nu_i = \begin{cases} \lambda_i^m q_m + y C_i, & y \in \mathbb{R}^{T_i}, \text{ if } i \in N'_m, \quad m = 1, \dots, k, \\ y C_i, & y \in \mathbb{R}^{T_i}, \quad \text{if } i \in N''_m, \quad m = 1, \dots, k, \end{cases} \quad (30)$$

$$(\beta_m)_i = \begin{cases} 0 & \text{if } i \in N'_m \setminus \{i_0\}, \quad m = 1, \dots, k-1, \\ \delta_i \mu & \text{if } i \in N'_k \setminus \{i_0\}, \quad m = k, \end{cases} \quad (31)$$

$$\varphi_i = 0, \quad i \in N, \quad \sigma = 0, \quad \|q_m\| = 1, \quad m = 1, \dots, k \}.$$

Here  $\mu$ ,  $\lambda_i^m$ , and  $y$  are free variables;

$\nu_i$  is an  $l$ -dimensional vector that corresponds to a fragment of agent  $i$ 's gradient of utility function related to his own consumption,  $i \in N$ ;

$\beta_m \in \mathbb{R}^{N_m \setminus \{i_0\}}$ ,  $m = 1, \dots, k$  correspond to the budget restrictions;

$\sigma \in \mathbb{R}^l$  reflects the market clearing condition;

$(q_1, \dots, q_k) \in \Theta$  corresponds to a hierarchic price representing non-standard equilibrium prices  $p$ ;

$\varphi_i \in \mathbb{R}^{T_i}$  relates to the condition  $x^i \in F^i$ .

Equations (30) are necessary conditions for the utility maximization problem under the restrictions imposed by the face  $F$  and the budget.

We present now a summary of results from Abraham and Robbin (1967) that will be used in the proof of the theorem.

**Definition 5 (transversality)** Let  $X$  and  $Y$  be  $C^1$  manifolds,  $f : X \rightarrow Y$  a  $C^1$  map and  $W \subseteq Y$  a submanifold. We say that  $f$  is *transversal to  $W$  at a point  $x \in X$*  (denoted by  $f \pitchfork_x W$ ), iff, where  $y = f(x)$ , either  $y \notin W$  or  $y \in W$  and

(1) the image  $T_x f(T_x X)$  contains a closed complement to  $T_y W$  in  $T_y Y$ , and

(2) the inverse image  $(T_x f)^{-1}(T_y W)$  splits (has a closed complement to  $T_x X$ ).

We say that  $f$  is *transversal to  $W$*  (in symbols  $f \pitchfork W$ ) iff  $f \pitchfork_x W$  for every  $x \in X$ .

Let  $\mathcal{V}, X, Y$  be  $C^r$  manifolds,  $\mathcal{C}^r(X, Y)$  the set of  $C^r$  maps from  $X$  to  $Y$  and  $\rho : \mathcal{V} \rightarrow \mathcal{C}^r(X, Y)$  a map. For  $v \in \mathcal{V}$  we write  $\rho_v$  instead of  $\rho(v)$ ; i.e.,  $\rho_v : X \rightarrow Y$  is a  $C^r$  map. We say  $\rho$  is a  *$C^r$  representation* if the evaluation map  $\omega_\rho : \mathcal{V} \times X \rightarrow Y$  given as  $\omega_\rho(v, x) = \rho_v(x)$  for  $v \in \mathcal{V}$  and  $x \in X$  is a  $C^r$  map from  $\mathcal{V} \times X$  to  $Y$ .

**Theorem 5 (Transversal Density Theorem)** *Let  $\mathcal{V}, X, Y$  be  $C^r$  manifolds,  $\rho : \mathcal{V} \rightarrow \mathcal{C}^r(X, Y)$  a  $C^r$  representation,  $W \subseteq Y$  a submanifold (not necessarily closed), and  $\omega_\rho : \mathcal{V} \times X \rightarrow Y$  the evaluation map. Assume that*

1.  $X$  has finite dimension  $n$  and  $W$  has finite codimension  $m$ ;
2.  $\mathcal{V}$  and  $X$  are second countable;
3.  $r > \max \{n - m, 0\}$ ;
4.  $\omega_\rho \pitchfork W$ .

*Then a set  $\mathcal{V}_W = \{v \in \mathcal{V} \mid \rho_v \pitchfork W\}$  is residual (and hence dense) in  $\mathcal{V}$ .*

**Theorem 6 (Openness of Transversal Intersection)** *Let  $\mathcal{V}, X, Y$  be  $C^r$  manifolds,  $\rho : \mathcal{V} \rightarrow \mathcal{C}^1(X, Y)$  a  $C^1$  representation,  $W \subseteq Y$  a  $C^1$  submanifold,  $K \subseteq X$  a compact subset, and*

1.  $X$  is of finite dimension;
2.  $W$  is closed.

*Then  $\mathcal{V}_{KW} = \{v \in \mathcal{V} \mid \rho_v \pitchfork_x W, \quad x \in K\}$  is open in  $\mathcal{V}$ .*

Denote by  $\mathcal{L}_i$  the linear hull of vectors  $\{c_t^i\}$ ,  $t \in T_i$  that constitute the rows of the matrix  $C_i$ .

**Lemma 3** *Consider an element  $(u_0, z_0) = (u_0, x_0, q_1^0, \dots, q_k^0) \in U \times Z$  such that*

$$q_m^0 \notin \mathcal{L}_i, \quad i \in N_m, \quad m = 1, \dots, k. \quad (32)$$

*Then  $\Psi : U \times Z \rightarrow \mathbb{R}^T \times \Theta$  is transversal at  $(u_0, z_0)$  to any submanifold of  $\mathbb{R}^T \times \Theta$ .*

**Proof.** It is sufficient to show that the tangent correspondence  $T_{(u_0, z_0)}\Psi$  is surjective. Let  $v = (v_1, \dots, v_n, v'_1, \dots, v'_k, v'', v''', v^1, \dots, v^n) \in \mathbb{R}^T \times \Theta$ . We need to find a differentiable path  $(u(\tau), z(\tau))_{\tau \in [0,1]}$  such that

$$u(0) = u_0, \quad z(0) = z_0 = (x_0, q_1^0, \dots, q_k^0), \quad \frac{\partial}{\partial \tau} \Psi(u(\tau), z(\tau))|_{\tau=0} = v.$$

Components  $x(\tau), q_m(\tau), m = 1, \dots, k$  of a path  $z(\tau)$  can be found in the following form:

$$x(\tau) = x_0 + \bar{x}\tau, \quad q_m(\tau) = q_m^0 + \bar{q}_m\tau, \quad m = 1, \dots, k, \quad (33)$$

where  $(\bar{x}, \bar{q}_1, \dots, \bar{q}_k)$  are determined from a system of linear equations

$$\begin{cases} (\sum_{i \in N} x^i(\tau))'|_{\tau=0} = v'', \\ (x^i(\tau), q_m(\tau))'|_{\tau=0} = v'_{mi}, \quad i \in N_m \setminus \{i_0\}, \quad m = 1, \dots, k, \\ (C_i x^i(\tau))'|_{\tau=0} = v^i, \quad i \in N, \\ (q_m(\tau))'|_{\tau=0} = v'''_m, \quad m = 1, \dots, k. \end{cases}$$

Substituting representation (33) into this system and taking the first derivatives at  $\tau = 0$  gives

$$\begin{cases} \sum_{i \in N} \bar{x}^i = v'', \\ (\bar{x}^i, q_m^0) = v'_{mi} - (x_0^i, \bar{q}_m), \quad i \in N_m \setminus \{i_0\}, \quad m = 1, \dots, k, \\ C_i \bar{x}^i = v^i, \quad i \in N, \\ \bar{q}_m = v'''_m, \quad m = 1, \dots, k. \end{cases}$$

Elements  $\bar{q}_1, \dots, \bar{q}_k$  can be found from the last  $kl$  equations. Take as  $\bar{x}$  any solution of a system

$$\begin{cases} \sum_{i \in N} \bar{x}^i = v'', \\ (\bar{x}^i, q_m^0) = \tilde{v}_{mi}, \quad i \in N_m \setminus \{i_0\}, \quad m = 1, \dots, k, \\ C_i \bar{x}^i = v^i, \quad i \in N, \end{cases} \quad (34)$$

where  $\tilde{v}_{mi} = v'_{mi} - (x_0^i, \bar{q}_m)$ . By the conditions of the lemma, all rows of a matrix  $D$  of this system are linearly independent, hence a solution exists.

Let us determine a path  $u(\tau)$  assuming  $u_i(\tau, x) = u_0(x) + b_i x^i \tau$ , where vectors  $b_i \in \mathbb{R}^l$  are chosen to satisfy

$$\frac{\partial}{\partial \tau} \left( \frac{\partial u_i}{\partial x_j^i}(x(\tau)) \right) \Big|_{\tau=0} = v_{ij}. \quad (35)$$

$$D = \begin{pmatrix} E & E & \cdots & E & \cdots & E & \cdots & E \\ 0 & q_1^0 & & & & & & \\ \vdots & & \ddots & & & & & \\ & & & q_1^0 & & & & \\ & & & & \ddots & & & \\ & & & & & q_k^0 & & \\ & & & & & & \ddots & \\ & & & & & & & q_k^0 \\ & C^i & & & & & & \\ & & \ddots & & & & & \\ & & & C^i & & & & \\ & & & & \ddots & & & \\ & & & & & C^i & & \\ \vdots & & & & & & \ddots & \\ 0 & & & & & & & C^i \end{pmatrix}$$

Figure 3: The matrix of the system (34).  $E$  is an  $(l \times l)$  identity matrix. First  $l$  columns of the matrix  $D$  correspond to the consumption of agent  $i_0$ .

Since  $x(\tau) = x_0 + \bar{x}\tau$ , the equation (35) transforms to

$$b_{ij} = v_{ij} - \sum_{s \in N \times L} \frac{\partial^2 (u_0)_i}{\partial x_j^i \partial x_s} (x_0) \bar{x}_s.$$

Since the vector  $v$  has been chosen arbitrarily, we have shown that the derivative mapping  $T_{(u_0, z_0)}\Psi$  is surjective. To complete the proof, we need to establish that  $(T_{(u_0, z_0)}\Psi)^{-1}(T_y W)$  splits. This follows from the surjectivity of  $T_{(u_0, z_0)}\Psi$  and from the finite dimensionality of its range.  $\square$

Linear dependence of the rows of a matrix  $\begin{pmatrix} q_m \\ C_i \end{pmatrix}$  or, which is the same, the condition

$$q_m \in \mathcal{L}_i, \quad i \in N_m \tag{36}$$

implies that the budget restriction  $q_m x^i = 0$  follows from the condition that the consumption plan  $x^i$  belongs to the face  $F_i$ . We are going now to remove the budget restrictions for agents from some set  $H \subset N$ , delete corresponding components  $(\Psi^m)_i$  of the mapping  $\Psi$  and accordingly restrict the domain of  $\Psi$

in the part concerning prices. The modified mapping  $\Psi^H$  will be transversal to any submanifold  $W \subseteq \mathbb{R}^l \times \Theta$  at any point  $(u, x, q_1, \dots, q_k) \in U \times \tilde{X} \times \Theta^H$  — the domain of  $\Psi^H$ .

Namely, let  $\Theta^H$  be a set of all elements  $(q_1, \dots, q_k)$  in  $\Theta$  such that  $q_m \in \mathcal{L}_i$  *only* for  $i \in H \cap N_m$ . Denote  $H \cap N_m$  by  $H_m$  and consider sets

$$W_m^H = \begin{cases} (\cap_{i \in H_m} \mathcal{L}_i) \setminus (\cup_{i \in N_m \setminus H_m} \mathcal{L}_i), & \text{if } H_m \neq \emptyset, \\ \mathbb{R}^l \setminus (\cup_{i \in N_m} \mathcal{L}_i) & \text{otherwise.} \end{cases} \quad (37)$$

Take sets

$$\Theta_m^H = \Theta^m \cap W_m^H, \quad m = 1, \dots, k$$

as components of the manifold  $\Theta^H$  :

$$\Theta^H = \Theta_1^H \times \dots \times \Theta_k^H.$$

Notice that sets  $\Theta^H$ ,  $H \subseteq N$  are relatively open and form a partition of the manifold  $\Theta$ .

Let  $\tilde{T} = T \setminus H$  and consider a mapping

$$\Psi_u^H = \prod_{i \in N} \Psi_i^u \times \Psi^{mc} \times \prod_{i \in N} \Psi_i^F \times \prod_{m=1}^k \Psi_H^m \times \Psi_H^q : \tilde{X} \times \Theta^H \rightarrow \mathbb{R}^{\tilde{T}} \times cl \Theta^H,$$

such that mappings  $\Psi_H^m : \tilde{X} \times \Theta_m^H \rightarrow \mathbb{R}^{N_m \setminus (H_m \cup \{i_0\})}$  are defined by

$$(\Psi_H^m(x, q_m))_i = q_m x^i, \quad i \in N_m \setminus (H_m \cup \{i_0\}),$$

and  $\Psi_H^q(q_1, \dots, q_k) = (q_1, \dots, q_k)$  is an identity embedding from  $\Theta^H$  to  $cl \Theta^H$ . All other mappings are defined as before. Put  $\Psi_H(u, z) = \Psi_u^H(z)$ . The manifold  $\Delta_{HF}$  is the analogue of the manifold  $\Delta_F$  :

$$\Delta_{HF} \stackrel{def}{=} \{(\nu_1, \dots, \nu_n, \beta_1, \dots, \beta_k, \sigma, q, \varphi_1, \dots, \varphi_n) \in \mathbb{R}^T \times cl \Theta^H\}$$

$$\nu_i = \begin{cases} \lambda_i^m q_m + y C_i, & y \in \mathbb{R}^{T_i} \quad \text{if } i \in N'_m \setminus H_m, \quad m = 1, \dots, k, \\ y C_i, & y \in \mathbb{R}^{T_i} \quad \text{otherwise,} \end{cases} \quad (38)$$

$$(\beta_m)_i = \begin{cases} 0 & \text{if } i \in N'_m \setminus (H_m \cup \{i_0\}), \quad m = 1, \dots, k-1, \\ \delta_i \mu & \text{if } i \in N'_k \setminus (H_k \cup \{i_0\}), \quad m = k, \end{cases} \quad (39)$$

$$\varphi_i = 0, \quad i \in N, \quad \sigma = 0, \quad \|q_m\| = 1, \quad m = 1, \dots, k\}.$$

**Lemma 4** *Suppose that  $u \in U$  is such that for every  $H \subseteq N$ ,  $\Theta$ , every partition  $(N_1, \dots, N_k)$  of the set  $N$ , and for all possible choices of subsets of satiated agents  $N_1'', \dots, N_k''$ , the mapping  $\Psi_u^H$  is transversal to the manifold  $\Delta_{HF}$ . Then the number of proper  $\delta$ -equilibria that belong to the face  $F$  is finite.*

**Proof.** The finiteness of proper  $\delta$ -equilibria follows from the finiteness (with respect to a choice of parameters  $\Theta, N_1, \dots, N_k, N_1'', \dots, N_k''$ , and  $H \subset N$ ) of the possibly different arrangements of the correspondence  $\Psi_u^H$  and the manifold  $\Delta_{HF}$ , as well as from the discreteness of the inverse images  $(\Psi_u^H)^{-1}(\Delta_{HF})$ , whose union covers all proper  $\delta$ -equilibria from  $F$ . Therefore, it will suffice to establish finiteness of the sets  $(\Psi_u^H)^{-1}(\Delta_{HF})$  to prove the lemma. By a well-known property of transversal correspondences,

$$\text{codim } \Delta_{HF} = \text{codim } (\Psi_u^H)^{-1}(\Delta_{HF}) \quad (40)$$

We show that

$$\text{codim } \Delta_{HF} \geq \dim \tilde{X} \times \Theta^H = nl + \dim \Theta^H. \quad (41)$$

Consider first only such equilibria that  $N_1 = N_1'$  (i. e., there is no satiation in the usual sense). By the construction and the assumptions made,  $\text{codim } \Delta_{HF}$  is equal to the difference between a number of restrictions and a number of free variables. Each restriction of type (39) or  $\varphi_i = 0$ ,  $i \in N$  corresponds to the free variable  $\lambda$  or  $y$  respectively. The budget restriction  $q_1 x^{i_0} = 0$  is omitted ( $i_0$  necessarily belongs to the set  $N_1$ ). Taking into account relations (38),  $\sigma = 0$ ,  $\|q_m\| = 1$ ,  $m = 1, \dots, k$ , and a free variable  $\mu$ , one gets

$$\text{codim } \Delta_{HF} = nl + l + k - 2.$$

Since  $\dim \Theta^H \leq l$

$$\text{codim } \Delta_{HF} \geq nl + \dim \Theta^H \quad (42)$$

if  $k > 1$ . Note, that  $\text{codim } \Delta_{HF}$  does not depend on the choice of  $H$ . If  $k = 1$  then  $N = N_1$ , so we can put the free variable  $\mu$  equal to zero (there are no satiated agents). This increases the codimension of the manifold by 1, and (42) is established again. In the case  $N_1'' \neq \emptyset$  we proceed in the same way with the only difference that  $\mu$  is not equated to zero but expressed through the values of the budget correspondences of the satiated agents.

By transversality of  $\Psi_u^H$  and relative openness of  $\Theta_m^H$  in  $\cap_{i \in H_m} \mathcal{L}_i$ , we conclude that

$$\begin{aligned} \dim (\Psi_u^H)^{-1}(\Delta_{HF}) &< 0, \text{ if } \dim \Theta^H \neq l; \\ \dim (\Psi_u^H)^{-1}(\Delta_{HF}) &= 0, \text{ if } \dim \Theta^H = l. \end{aligned}$$

Thus,  $(\Psi_u^H)^{-1}(\Delta_{HF})$  is discrete whenever  $\dim \Theta^H = l$ ,  $k \leq 2$  or is empty otherwise. One can easily see that there is only one subset  $\bar{H} \subseteq N$  such that  $\dim \Theta^{\bar{H}} = l$ .

Since  $\Delta_F$  is closed, and  $X$  is, without the loss of generality, compact in  $\mathbb{R}^{ln}$ , the intersection  $(\Psi_u)^{-1}(\Delta_F) \cap (X \times \Theta)$  is compact in  $\mathbb{R}^{ln} \times \Theta$ . We have the following relation

$$(\Psi_u)^{-1}(\Delta_F) \subseteq \bigcup_H (\Psi_u^H)^{-1}(\Delta_{HF}) = (\Psi_u^{\bar{H}})^{-1}(\Delta_{HF}).$$

But  $(\Psi_u^{\bar{H}})^{-1}(\Delta_{HF})$  is discrete in  $\tilde{X} \times \Theta^H$ . Therefore,  $(\Psi_u)^{-1}(\Delta_F) \cap (X \times S)$  is a discrete compact, which implies the finiteness of proper  $\delta$ -equilibria from  $F$ .

□

**Proof of Theorem 4.** For each  $i \in N$  choose a compact  $K_i \in \mathbb{R}^{L \times \{i\}}$  such that  $\text{int } K_i \supset X_i$ ,  $K = \prod_{i \in N} K_i \subset \tilde{X}$ , and let  $\{S_t^H\}_{t=1}^\infty$  be a sequence of compact sets approximating  $\Theta^H$  from within:

1.  $S_t^H \subset \Theta^H$ ,  $t = 1, 2, \dots$ ,
2.  $\bigcup_{t=1}^\infty S_t^H = \Theta^H$ .

Consider a sequence of compact sets  $K_H^t = K \times S_t^H \subset \tilde{X} \times \Theta^H$ ,  $t = 1, 2, \dots$ , and apply the theorems of density and openness of transversal sections to the case  $\mathcal{V} = U$ ,  $X = \tilde{X} \times \Theta^H$ ,  $Y = \mathbb{R}^T \times \text{cl } \Theta^H$ ,  $\rho_v = \Psi_u^H$ ,  $\omega_\rho = \Psi_H$ ,  $W = \Delta_{HF}$ . By construction,  $\Psi_H$  is transversal to  $\Delta_{HF}$ , and all other conditions of Thom's theorems are satisfied as well. Therefore, the set  $\mathcal{V}_{KW}^t = \{u \in U \mid \Psi_u^H \pitchfork_z \Delta_{HF}, z \in K_H^t\}$  is open and dense in  $U$ . Let

$$G = \bigcap_{t=1}^\infty \mathcal{V}_{KW}^t,$$

where the intersection is taken over  $t = 1, 2, \dots$  and over all admissible  $F$ ,  $\Theta^H$ ,  $H$ ,  $N'_1, \dots, N'_k$ ,  $i_0$ . Since  $G$  is a countable intersection of open dense sets, it is residual. Direct application of Lemma 4 completes the proof.

□

## 8 Appendix

**Proof of Proposition 2.** It is sufficient to show that  $B_i(\bar{p}) = \bar{B}_i(p)$  for every  $i \in N$ . For a fixed  $i$ , assume  $x \in \bar{B}_i(p)$  and let  $(q_1, \dots, q_k)$  be a



hierarchical representation of the price vector  $p$ . Note that  $\bar{p} = {}^\circ(p/\|p\|) = q_1$ . There exists  $\tilde{x} \in {}^*X_i$  such that  $p\tilde{x} \leq pw^i$  and  $\tilde{x} \approx x$ . Assume inequality  $q_1x > q_1w^i$ . Then

$$q_1\tilde{x} + \frac{\lambda_2}{\lambda_1}q_2\tilde{x} + \cdots + \frac{\lambda_k}{\lambda_1}q_k\tilde{x} > q_1w^i + \frac{\lambda_2}{\lambda_1}q_2w^i + \cdots + \frac{\lambda_k}{\lambda_1}q_kw^i,$$

which yields a contradiction with  $p\tilde{x} \leq pw^i$ . Therefore  $q_1x \leq q_1w^i$  and  $x \in B_i(\bar{p})$ .

Let  $x \in B_i(\bar{p})$  and find  $y \in B_i(\bar{p})$  such that  $\bar{p}y < \bar{p}w^i$ . Consider an internal sequence  $\alpha : {}^*N \rightarrow {}^*X_i$  defined by  $\alpha(n) = (1/n)y + (1-1/n)x$ . Since  $p\alpha(n) < pw^i$  for all  $n \in N$ , it should be also true for some hyperfinite natural number  $\tilde{n} \in {}^*N \setminus N$ . Then  $\alpha(\tilde{n}) \approx x$ , which implies  $x \in st {}^*B_i(p) = \bar{B}_i(p)$ . We have shown that  $B_i(\bar{p}) = \bar{B}_i(p)$  for each  $i \in N$ . Hence  $\bar{x}$  is a Walrasian equilibrium with equilibrium prices  $\bar{p}$ .

□

## Hierarchical equilibria

We are going to give a formal proof of the equivalence between non-standard dividend equilibria and hierarchical equilibria for an exchange economy with polyhedral consumption sets. This follows easily from Proposition 6 that provides the equivalence between budget sets given by non-standard prices and dividends and by hierarchical prices and revenues. In fact, to prove the equivalence between those two sorts of equilibria, one needs only to point out the relation between the terms in which they were described.

Consider a hierarchical price  $q = (q_1, \dots, q_k)$  and define the  $q$ -value of a consumption bundle  $x$  as a vector

$$qx = (q_1x, \dots, q_kx, +\infty, \dots, +\infty) \in (\mathbb{R} \cup \{+\infty\})^l.$$

A *hierarchical revenue*  $r$  is an element of  $(\mathbb{R} \cup \{+\infty\})^l$ . For a hierarchical price  $q$ , revenue  $r$  and  $i \in N$  consider

$$s_i(q, r) = \min \{s \in \{1, \dots, l\} \mid \exists x \in X_i : (q_1x, \dots, q_sx) \prec (r_1, \dots, r_s)\},$$

where  $\prec$  denotes a lexicographic ordering. In principle,  $s_i$  is the first level at which a consumer is not at minimum wealth. Given  $s_i(q, r)$ , consider an augmented revenue vector

$$\rho_i(q, r) = (r_1, \dots, r^{s_i(q, r)}, +\infty, \dots, +\infty) \in (\mathbb{R} \cup \{+\infty\})^l$$

and the budget set of consumer  $i$

$$B_i(q, r) = \{x \in X_i \mid qx \preceq \rho_i(q, r)\}.$$

This construction guarantees closedness of the budget set  $B_i(q, r)$ .

**Definition 6** A list of net trades  $x \in \prod_{i \in N} (X_i - w^i)$  is a *hierarchical equilibrium* of the economy  $\mathcal{E}$  if there exist a hierarchical price  $q$  and positive hierarchical revenues  $r^i \in \mathbb{R}_+^l$ ,  $i \in N$  such that:

- (i)  $x^i \in B_i(q, r^i)$  and  $\mathcal{P}_i(x) \cap B_i(q, r^i) = \emptyset$ ,  $i \in N$ ;
- (ii)  $\sum_{i \in N} x^i = 0$ .

**Theorem 7** Suppose that consumption sets  $X_i$  of all individuals are polyhedral. Then the set of hierarchical equilibria of an economy  $\mathcal{E}$  coincides with the set of non-standard dividend equilibria.

**Proof.** Suppose that  $x$  is a hierarchical equilibrium, and that  $q$  and  $r$  are the corresponding hierarchical price and revenue respectively. Then each consumer maximizes his preferences on the set

$$B_i(q, r^i) = \{x \in X_i(q_1, \dots, q_{s_i-1}) \mid q_{s_i} x \leq r_{s_i}\},$$

which immediately implies that  $x$  is a non-standard dividend equilibrium at prices  $p = q_1 + \varepsilon q_2 + \dots + \varepsilon^{k-1} q_k$  for some  $\varepsilon > 0$ ,  $\varepsilon \approx 0$ , and dividends

$$d_i = \varepsilon^{s_i-1} r_{s_i}, \quad i \in N.$$

Conversely, if  $x$  is a non-standard dividend equilibrium at  $p \in {}^*\mathbb{R}^l$  and  $d \in \mathbb{R}_+^n$ , then the representation  $q^p$  will be a hierarchical equilibrium price, and for each  $i \in N$  the components of a hierarchical equilibrium revenue  $r^i$  will be

$$r_t^i = \begin{cases} 0 & \text{if } t < j(p, d_i) \\ \circ(d_i / \lambda_{j(p, d_i)}) & \text{if } t = j(p, d_i) \\ +\infty & \text{if } t > j(p, d_i). \end{cases}$$

□

## 9 References

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